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EXAMPLES

OF THE PROCESSES OF THE

DIFFERENTIAL AND INTEGRAL CALCULUS.

COLLECTED BY

D. F. GREGORY, M.A.,

FELLOW OF TRINITY COLLEGE.

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PREFACE.

THE chief object of the present work is, as its title indicates, to furnish to the student examples by which to illustrate the processes of the Differential and Integral Calculus. In this respect it will be seen to agree with Professor Peacock's Collection of Examples; and indeed if a second edition of that excellent work had been published I should not have undertaken the task of making this compilation. But as Professor Peacock informed me that he had not leisure to superintend the publication of a second edition of his "Examples" which had been long out of print, I thought that I should do a service to students by preparing a work on a similar plan, but with such modifications as seemed called for by the increased cultivation of Analysis in this University. Accordingly I have not limited myself to the mere collection of Examples and Problems illustrative of Theorems given in Elementary Treatises on the subject, but I have also introduced demonstrations of propositions which, although important and interesting, do not usually find a place in works devoted to the exposition of the principles of the Calculus. I wished by these means to render this Collection, as it were, complementary to those works, and, with the view of allowing it to be read in connection with any of them, I have generally assumed as known only those methods which are to be found in all Elementary Treatises. To this, however, there is one exception: it will be seen that I have made constant use of the method known by the name of the Separation of the Symbols of Operation, although the Theory of the process is not usually given in works which are likely to be in the hands of students. I have done so because I think it a matter of some importance that the use of this method should be extended as much as possible, since it shortens and simplifies many of the processes of the Calculus, while at the same time it offers to the student one of the most instructive examples of Analytical Generalization. There seems to have been among writers on the Calculus an unwillingness to consider this method in any other light than as founded on an accidental analogy, and therefore to reject it as not based on a strict logical deduction. This idea I think is formed on a limited view of the nature of Analysis, and I shall be glad if the use which I have made of the Separation of the Symbols may induce others to examine the question closely, and so satisfy themselves of the logical validity of the process. The principles of the method are so simple that I think the short sketch which I have given of them in Chap. XV. will be sufficient to make its application readily understood.

I have adhered throughout to the notation of Leibnitz in preference to that which has been of late revived and partially adopted in this University. Of the Differential notation I need say nothing here, as it appears to be abandoned as an *exclusive* system by those who introduced it: but as the use of the suffix notation for integrals has been sanctioned by those whose names are of high authority, I may state briefly some of my reasons for differing from them. In the first place, on considering the subject, I could find no arguments against the use of the notation for Differentials, which did not apply with even greater force against that for integrals: indeed, although there may be some cases in which the use of the former is advantageous, I know of none in which the latter does not appear to me to be inconvenient. In the next place, I fully agree with Professor De Morgan in an unwillingness to lose sight of the analogy to summation which is implied in the old notation; and if it were at any time necessary to consider integration merely as the inverse of differentiation, I should prefer to employ such a symbol as d_x^{-1} which expresses the required idea better than $\int_x$. But what I look on as a fatal objection to the suffix integral notation is that, like the corresponding one for differentials, it is not applicable to all cases. Of this any one may satisfy himself by attempting to use it in transforming a multiple Integral from one system of independent variables to another, a problem which is of frequent occurrence, but which I have not seen solved analytically in any work in which the suffix notation is employed. So long, therefore, as the old notation adapts itself to all cases in which it is required, while that which is proposed is not so accommodating, there appears to me no doubt which is to be preferred.

The sources from which the Examples have been taken are indicated by the references which will be found in the body of the work. For although I have not thought it necessary to cite an authority for every example, I have done so in all cases in which the student would be likely to wish for more information by consulting the original authors. It has always appeared to me that we sacrifice many of the advantages and more of the pleasures of studying any science by omitting all reference to the history of its progress: I have therefore occasionally introduced historical notices of those problems which are interesting either from the nature of the questions involved, or from their bearing on the history of the Calculus. From a fear of increasing the size of the volume too much, I have not done this to as great an extent as I wished, but these digressions short as they are may serve to relieve the dryness of a mere collection of Examples.

Trinity College,
October, 1841.

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DIFFERENTIAL CALCULUS.

CHAPTER I.

DIFFERENTIATION.

Functions of One Variable.

If u be an explicit function of x, which is of a complicated form, it may generally be reduced to the differentiation of simpler functions by means of the theorem

$$\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx},$$

y being some function of x, and u some function of y. This theorem may be extended to any number of functions, so that

$$\frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dz} \cdot \frac{dz}{dy} \cdots\cdots \frac{dy}{dx}.$$

Ex. (1) Let $u = (a + bx^n)^m$.

Then $y = a + bx^n, \qquad u = y^m,$

$$\frac{dy}{dx} = nbx^{n-1}, \quad \frac{du}{dy} = my^{m-1} = m(a + bx^n)^{m-1};$$

$$\text{therefore } \frac{du}{dx} = mnbx^{n-1}(a + bx^n)^{m-1}.$$

(2) $$u = \{x + (1 + x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}, \qquad \frac{du}{dx} = \frac{\{x + (1 + x^2)^{\frac{1}{2}}\}^{\frac{1}{2}}}{2(1 + x^2)^{\frac{1}{2}}}.$$

(3) $$u = \epsilon^{x^n}; \quad \frac{du}{dx} = n x^{n-1} \epsilon^{x^n}.$$

(4) $$u = \epsilon^{\sin x}; \quad \frac{du}{dx} = \cos x \, \epsilon^{\sin x}.$$

(5) $$u = \log \{x + (1 + x^2)^{\frac{1}{2}}\}; \quad \frac{du}{dx} = \frac{1}{(1 + x^2)^{\frac{1}{2}}}.$$

(6) $$u = \log (\log x) = \log^2 (x); \quad \frac{du}{dx} = \frac{1}{x \log x}.$$

(7) $u = \log^n x$; which signifies not the n^{th} power of the logarithm of x, but the n^{th} logarithm of that quantity,

$$\frac{du}{dx} = \frac{1}{x \log x \log^2 x \ldots\ldots \log^{n-1} x}.$$

(8) $$u = \log (\sin x); \quad \frac{du}{dx} = \cot x.$$

(9) $$u = \log \left(\frac{1 - \cos m x}{1 + \cos m x}\right)^{\frac{1}{2}}; \quad \frac{du}{dx} = \frac{m}{\sin m x}.$$

(10) $$u = \log (\tan x), \quad \frac{du}{dx} = \frac{2}{\sin 2x}.$$

(11) $$u = \cos (\sin x); \quad \frac{du}{dx} = - \cos x \sin^2 x,$$

$\sin^2 x$ being the same as $\sin (\sin x)$.

(12) $$u = \sin (\log x), \quad \frac{du}{dx} = \frac{1}{x} \cos (\log x).$$

(13) $$u = \sin^{-1} \frac{x}{(1 + x^2)^{\frac{1}{2}}}, \quad \frac{du}{dx} = \frac{1}{1 + x^2}.$$

(14) $$u = \sin^{-1} \frac{1 - x^2}{1 + x^2}, \quad \frac{du}{dx} = - \frac{2}{1 + x^2}.$$

(15) $$u = \epsilon^{\sin^{-1} x}; \quad \frac{du}{dx} = \frac{1}{(1 - x^2)^{\frac{1}{2}}} \epsilon^{\sin^{-1} x}.$$

(16) $$u = \cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right),$$

$$\frac{du}{dx} = \frac{(a^2 - b^2)^{\frac{1}{2}}}{a + b\cos x}.$$

(17) $$u = \sin^{-1}\frac{x^2}{a^2}; \quad \frac{du}{dx} = \frac{2x}{(a^4 - x^4)^{\frac{1}{2}}}.$$

(18) $$u = \sin^{-1}\left(\frac{x^2 - a^2}{b^2 - a^2}\right)^{\frac{1}{2}},$$

$$\frac{du}{dx} = \frac{x}{\{(x^2 - a^2)(b^2 - x^2)\}^{\frac{1}{2}}}.$$

(19) $$u = \sin^{-1}\frac{x - 1}{2^{\frac{1}{2}}}, \quad \frac{du}{dx} = \frac{1}{(1 + 2x - x^2)^{\frac{1}{2}}}.$$

(20) $$u = \tan^{-1}\{(1 + x^2)^{\frac{1}{2}} - x\}, \quad \frac{du}{dx} = -\frac{1}{2(1 + x^2)}.$$

(21) $$u = \tan^{-1}\frac{2x}{1 - x^2}, \quad \frac{du}{dx} = \frac{2}{1 + x^2}.$$

(22) $$u = \tan^{-1}\frac{2cx + b}{(4ac - b^2)^{\frac{1}{2}}}, \quad \frac{du}{dx} = \frac{1}{2}\,\frac{(4ac - b^2)^{\frac{1}{2}}}{a + bx + cx^2}.$$

(23) $$u = \tan^{-1}\left(\frac{a + bx}{b - a}\right)^{\frac{1}{2}},$$

$$\frac{du}{dx} = \frac{1}{2}\,\frac{(b - a)^{\frac{1}{2}}}{(1 + x)(a + bx)^{\frac{1}{2}}}.$$

(24) $$u = \sin^{-1}\frac{x(a - b)^{\frac{1}{2}}}{\{a(1 + x^2)\}^{\frac{1}{2}}},$$

$$\frac{du}{dx} = \frac{(a - b)^{\frac{1}{2}}}{(1 + x^2)(a + bx^2)^{\frac{1}{2}}}.$$

(25) $$u = \log\frac{\{(1 + x^2)^{\frac{1}{2}} + x\,2^{\frac{1}{2}}\}}{(1 - x^2)^{\frac{1}{2}}},$$

$$\frac{du}{dx} = \frac{2^{\frac{1}{2}}}{(1 - x^2)(1 + x^2)^{\frac{1}{2}}}.$$

(26) $$u = \log\left\{x + (x^2 - a^2)^{\frac{1}{2}}\right\} + \sec^{-1}\frac{x}{a},$$

$$\frac{du}{dx} = \frac{1}{x}\left(\frac{x+a}{x-a}\right)^{\frac{1}{2}}.$$

(27) $$u = \cos^{-1} x - 2\,\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}, \quad \frac{du}{dx} = \frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{3}{2}}}.$$

(28) $$u = \frac{\sin x\,(2 + e\cos x)}{(1 + e\cos x)^2},$$

$$\frac{du}{dx} = \frac{3e + (2 + e^2)\cos x}{(1 + e\cos x)^3}.$$

(29) $$u = \left\{\sin(a^2 - x^2)\right\}^{\frac{1}{2}}, \quad \frac{du}{dx} = -\frac{x\cos(a^2 - x^2)}{\left\{\sin(a^2 - x^2)\right\}^{\frac{1}{2}}}.$$

(30) $$u = \log\cos^{-1}(1 - x^2)^{\frac{1}{2}}, \quad \frac{du}{dx} = \frac{1}{(1 - x^2)^{\frac{1}{2}}\sin^{-1} x}.$$

When a function consists of products and quotients of roots and powers, it is generally most convenient to take the differential of the logarithm, or, as it is usually called, the logarithmic differential of the function.

(31) Let $u = (a + x)^m (b + x)^n$,

$$\log u = m\log(a + x) + n\log(b + x),$$

$$\frac{1}{u}\frac{du}{dx} = \frac{m}{a + x} + \frac{n}{b + x},$$

$$\frac{du}{dx} = (a + x)^m (b + x)^n \left(\frac{m}{a + x} + \frac{n}{b + x}\right).$$

(32) $$u = \left(\frac{x - 1}{x + 1}\right)^{\frac{1}{2}},$$

$$\frac{du}{dx} = \left(\frac{x - 1}{x + 1}\right)^{\frac{1}{2}} \frac{1}{x^2 - 1} = \frac{1}{(x - 1)^{\frac{1}{2}}(x + 1)^{\frac{3}{2}}}.$$

(33) $$u = \frac{x^n}{(1 + x)^n}, \quad \frac{du}{dx} = \frac{nx^{n-1}}{(1 + x)^{n+1}}.$$

(34) $$u = \frac{(x-2)^9}{\{(x-1)^5 (x-3)^{11}\}^{\frac{1}{2}}},$$

$$\frac{du}{dx} = \frac{(x-2)^8}{(x-1)^{\frac{7}{2}} (x-3)^{\frac{13}{2}}} (x^2 - 7x + 1).$$

(35) $$u = \frac{(x+4)^2}{x+2}, \quad \frac{du}{dx} = \frac{x(x+4)}{(x+2)}.$$

(36) $$u = \frac{\{(x+1)(x+3)^9\}^{\frac{1}{2}}}{(x+2)^4},$$

$$\frac{du}{dx} = \frac{x^2}{(x+2)^5} \left\{\frac{(x+3)^7}{x+1}\right\}^{\frac{1}{2}}.$$

(37) $$u = x^x, \quad \log u = x \log x,$$

$$\frac{du}{dx} = x^x (1 + \log x).$$

(38) $$u = x^{\sin x}; \quad \frac{du}{dx} = x^{\sin x} \left(\cos x . \log x + \frac{\sin x}{x}\right).$$

(39) $$u = (\sin x)^m (\cos x)^n,$$

$$\frac{du}{dx} = (\sin x)^{m-1} (\cos x)^{n-1} (m \cos^2 x - n \sin^2 x).$$

(40) $$u = \frac{(\sin x)^m}{(\cos x)^n},$$

$$\frac{du}{dx} = \frac{(\sin x)^{m-1}}{(\cos x)^{n+1}} (m \cos^2 x + n \sin^2 x).$$

(41) $$u = \epsilon^{ax} \sin rx, \quad \frac{du}{dx} = \epsilon^{ax} (a \sin rx + r \cos rx),$$

$$u = \epsilon^{ax} \cos rx, \quad \frac{du}{dx} = \epsilon^{ax} (a \cos rx - r \cos rx),$$

(42) $$u = \epsilon^{ax} (\sin rx)^m,$$

$$\frac{du}{dx} = \epsilon^{ax} (\sin rx)^{m-1} (a \sin rx + mr \cos rx).$$

Implicit Functions of Two Variables.

If $u = 0$ be an implicit function of two variables x and y, then

$$\frac{dy}{dx} = -\frac{\dfrac{du}{dx}}{\dfrac{du}{dy}}.$$

(43) Let $x \log y = y \log x$;

then $$\frac{dy}{dx} = \frac{y}{x}\left(\frac{y - x \log y}{x - y \log x}\right).$$

(44) If $\sin y = x \sin (a + y)$,

$$\frac{dy}{dx} = \frac{\sin (a + y)}{\cos y - x \cos (a + y)}.$$

(45) If $y^n \log y = ax$,

$$\frac{dy}{dx} = \frac{a}{y^{n-1} (1 + n \log y)}.$$

(46) If $\tan y = 1 + x \sin y$,

$$\frac{dy}{dx} = \frac{(\cos y)^2 \sin y}{1 - x (\cos y)^3}.$$

(47) Let $\tan \dfrac{y}{2} = \left(\dfrac{1 - x}{1 + x}\right)^{\frac{1}{2}}$;

taking the logarithmic differential we find

$$\frac{dy}{dx} = -\frac{\sin y}{1 - x^2} = -\frac{1}{(1 - x^2)^{\frac{1}{2}}}.$$

(48) If $y = 1 + x\epsilon^y$,

$$\frac{dy}{dx} = \frac{\epsilon^y}{1 - x\epsilon^y} = \frac{\epsilon^y}{2 - y}.$$

(49) Let $x(1+y)^{\frac{1}{2}} + y(1+x)^{\frac{1}{2}} = 0$;

then $$\frac{dy}{dx} = \frac{y}{x} \cdot \frac{y + 2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}{x + 2(1+x)^{\frac{1}{2}}(1+y)^{\frac{1}{2}}}.$$

(50) Let $\sin^{-1}\frac{x}{h} + \sin^{-1}\frac{y}{k} = c$;

then $$\frac{dy}{dx} = -\frac{(k^2-y^2)^{\frac{1}{2}}}{(h^2-x^2)^{\frac{1}{2}}}.$$

(51) Let $(x^2+y^2)^2 = a^2x^2 - b^2y^2$,

$$\frac{dy}{dx} = \frac{\{a^2 - 2(x^2+y^2)\}\, x}{\{b^2 + 2(x^2+y^2)\}\, y}.$$

(52) Let $(a+y)^2(b^2-y^2) - x^2y^2 = 0$,

then $$\frac{dy}{dx} = -\frac{y^2(b^2-y^2)^{\frac{1}{2}}}{y^3 + ab^2}.$$

Functions of Two or more Variables.

(53) $$u = \left(\frac{x^2-y^2}{x^2+y^2}\right)^{\frac{1}{2}},$$

$$\frac{du}{dx} = \frac{2xy^2}{(x^2+y^2)^{\frac{3}{2}}(x^2-y^2)^{\frac{1}{2}}},\quad \frac{du}{dy} = -\frac{2x^2y}{(x^2+y^2)^{\frac{3}{2}}(x^2-y^2)^{\frac{1}{2}}},$$

$$du = \frac{2xy(y\,dx - x\,dy)}{(x^2+y^2)^{\frac{3}{2}}(x^2-y^2)^{\frac{1}{2}}}.$$

(54) $$u = \frac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x+y},$$

$$\frac{du}{dx} = \frac{y - x - 2(xy)^{\frac{1}{2}}}{2x^{\frac{1}{2}}(x+y)^2},\quad \frac{du}{dy} = \frac{x - y - 2(xy)^{\frac{1}{2}}}{2y^{\frac{1}{2}}(x+y)^2},$$

$$du = \frac{\{y-x-2(xy)\}^{\frac{1}{2}}y^{\frac{1}{2}}dx + \{x-y-2(xy)\}^{\frac{1}{2}}x^{\frac{1}{2}}dy}{2(xy)^{\frac{1}{2}}(x+y)^2}.$$

(55) $$u = x^y, \quad \frac{du}{dx} = y x^{y-1}, \quad \frac{du}{dy} = x^y \log x,$$

$$du = x^y \left(\frac{y}{x} dx + \log x dy\right).$$

(56) $$u = \log \left\{\frac{x + (x^2 - y^2)^{\frac{1}{2}}}{x - (x^2 - y^2)^{\frac{1}{2}}}\right\},$$

$$\frac{du}{dx} = \frac{2y}{y(x^2 - y^2)^{\frac{1}{2}}}, \quad \frac{du}{dy} = -\frac{2x}{y(x^2 - y^2)^{\frac{1}{2}}},$$

$$du = \frac{2(ydx - xdy)}{y(x^2 - y^2)^{\frac{1}{2}}}.$$

(57) Let $u = \sin(x^m y^n)$,

$$du = x^{m-1} y^{n-1} \cos(x^m y^n)(my dx + nx dy).$$

(58) If $u = \sin^{-1}\frac{x}{y}$, $du = \frac{ydx - xdy}{y(y^2 - x^2)^{\frac{1}{2}}}$.

(59) If $u = \tan^{-1}\frac{x}{y}$, $du = \frac{ydx - xdy}{x^2 + y^2}$.

(60) If $u = \log\left(\tan\frac{x}{y}\right)$, $du = \frac{2(ydx - xdy)}{x^2 \sin 2\frac{x}{y}}$.

(61) If $u = \frac{\epsilon^z y}{(x^2 + y^2)^{\frac{1}{2}}}$,

$$du = \frac{\epsilon^z y dz}{(x^2 + y^2)^{\frac{1}{2}}} + \frac{x\epsilon^z (xdy - ydx)}{(x^2 + y^2)^{\frac{3}{2}}}.$$

(62) If $u = \frac{x^2 y}{a^2 - z^2}$,

$$du = \frac{2xydx}{a^2 - z^2} + \frac{x^2 dy}{a^2 - z^2} + \frac{2x^2 yz dz}{(a^2 - z^2)^2}.$$

(63) $$u = (x^2 + y^2 + z^2)^{\frac{1}{2}} + \tan^{-1}\frac{x}{z} + \frac{z^2}{2}.$$

$$du = \frac{x\,dx + y\,dz + z\,dz}{(x^2 + y^2 + z^2)^{\frac{1}{2}}} + \frac{z\,dx - x\,dz}{x^2 + z^2} + z\,dz.$$

(64) $$u = \frac{ay - bz}{cz - ax},$$

$$du = \frac{a}{(cz - ax)^2}\left\{(ay - bz)\,dx + (cz - ax)\,dy + (bx - cy)\,dz\right\}.$$

CHAPTER II.

SUCCESSIVE DIFFERENTIATION.

THE analogy between Algebraic powers and successive differentials, when expressed by the notation of Leibnitz, was observed soon after the invention of the Calculus. Leibnitz himself paid much attention to this subject, as may be seen in his correspondence with John Bernoulli; and, in the course of his investigations, he discovered, by induction, the Theorem which bears his name. He also conceived the existence of differentials with fractional or irrational indices, but he made no steps towards the calculation of such functions in any cases. In recent years that branch of the Calculus has acquired considerable importance, and it appears to be the quarter from which we may look for great additions to our knowledge of analysis. I shall however in this chapter confine myself to examples of differentiation with integer indices, partly because there are still some points in the theory of general differentiation which are not entirely fixed, so that the subject is not adapted for the student; partly because the principles of that branch of the Calculus are not laid down in any Elementary Treatises which a student could consult, and it would occupy too much space to enter at large on the subject in the following pages. Those who wish to see the results of the labours of mathematicians in this field of research are referred to various Memoirs of Liouville in the *Journal de l'Ecole Polytechnique*, Vol. XIII., and in *Crelle's Journal;* to two papers by Professor Kelland in the *Transactions of the Royal Society of Edinburgh*, Vol. XIV.; to Professor Peacock's *Report on the Progress of Analysis* in the Transactions of the British Association; and to two papers by Mr Greatheed in the *Cambridge Mathematical Journal*, Vol. I.

SECT. I. *Functions of One Variable.*

(1) $u = x^n$; $\dfrac{d^r u}{dx^r} = n(n-1) \ldots\ldots (n-r+1)\, x^{n-r}$.

(2) $u = (a+bx)^n$; .

$$\frac{d^r u}{dx^r} = n(n-1) \ldots\ldots (n-r+1)\, b^r (a+bx)^{n-r}.$$

(3) $u = \dfrac{1}{x^n}$; $\dfrac{d^r u}{dx^r} = (-)^r n(n+1) \ldots (n+r-1) \dfrac{1}{x^{n+r}}$.

(4) $u = \dfrac{1}{x}$; $\dfrac{d^r u}{dx^r} = (-)^r r(r-1) \ldots 3 \,.\, 2 \,.\, 1 \,.\, \dfrac{1}{x^{r+1}}$.

(5) $u = a^x$; $\dfrac{d^r u}{dx^r} = (\log a)^r a^x$.

(6) $u = \epsilon^{nx}$; $\dfrac{d^r u}{dx^r} = n^r \epsilon^x$.

(7) $u = \sin nx$,

$$\frac{du}{dx} = n \cos nx = n \sin\left(nx + \frac{\pi}{2}\right),$$

$$\frac{d^2 u}{dx^2} = n \frac{d}{dx} \,.\, \sin\left(nx + \frac{\pi}{2}\right) = n^2 \cos\left(nx + \frac{\pi}{2}\right),$$

$$= n^2 \sin\left(nx + \frac{\pi}{2} + \frac{\pi}{2}\right) = n^2 \sin\left(nx + 2\frac{\pi}{2}\right).$$

By continuing the same process, we find

$$\frac{d^r u}{dx^r} = n^r \sin\left(nx + r\frac{\pi}{2}\right).$$

In the same way we have

(8) $u = \cos nx$; $\dfrac{d^r u}{dx^r} = n^r \cos\left(nx + r\dfrac{\pi}{2}\right)$.

(9) $u = \epsilon^{x\cos\theta}\cos(x\sin\theta)$;

$$\frac{du}{dx} = \epsilon^{x\cos\theta}\left\{\cos(x\sin\theta)\cos\theta - \sin(x\sin\theta)\sin\theta\right\}$$

$$= \epsilon^{x\cos\theta}\cos(x\sin\theta + \theta),$$

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\epsilon^{x\cos\theta}\cos(x\sin\theta + \theta)$$

$$= \epsilon^{x\cos\theta}\cos(x\sin\theta + \theta + \theta) = \epsilon^{x\cos\theta}\cos(x\sin\theta + 2\theta).$$

By continuing the same process, we find

$$\frac{d^ru}{dx^r} = \epsilon^{x\cos\theta}\cos(x\sin\theta + r\theta).$$

Murphy, *Cambridge Transactions*, Vol. v. p. 342.

(10) $u = \epsilon^{ax}\cos nx$,

$$\frac{du}{dx} = \epsilon^{ax}(a\cos nx - n\sin nx).$$

Let $\frac{n}{a} = \tan\phi$, so that

$$a = (a^2 + n^2)^{\frac{1}{2}}\cos\phi, \qquad n = (a^2 + n^2)^{\frac{1}{2}}\sin\phi.$$

Then $\frac{du}{dx} = (a^2 + n^2)^{\frac{1}{2}}\epsilon^{ax}(\cos\phi\cos nx - \sin\phi\sin nx)$

$$= (a^2 + n^2)^{\frac{1}{2}}\epsilon^{ax}\cos(nx + \phi).$$

Hence as before,

$$\frac{d^ru}{dx^r} = (a^2 + n^2)^{\frac{r}{2}}\epsilon^{ax}\cos(nx + r\phi);$$

Similarly, if $u = \epsilon^{ax}\sin nx$,

$$\frac{d^ru}{dx^r} = (a^2 + n^2)^{\frac{r}{2}}\epsilon^{ax}\sin(nx + r\phi).$$

(11) $u = \log x, \quad \frac{du}{dx} = \frac{1}{x},$

$$\frac{d^r u}{dx^r} = \frac{d^{r-1}}{dx^{r-1}} \frac{1}{x} = (-)^{r-1} (r-1)(r-2)\ldots 3 \,.\, 2 \,.\, 1 \frac{1}{x^r},$$

by Ex. 4.

(12) $u = \frac{1+x}{1-x}, \quad \frac{du}{dx} = \frac{2}{(1-x)^2},$

$$\frac{d^r u}{dx^r} = \frac{d^{r-1}}{dx^{r-1}} \frac{2}{(1-x)^2} = \frac{2 \,.\, r(r-1)\ldots 3 \,.\, 2}{(1-x)^{r+1}}.$$

In functions consisting of the product of two or more simple functions, we may make use of the Theorem of Leibnitz, the enunciation of which is as follows.

If u, v be two functions of x, then

$$\frac{d^r(uv)}{dx^r} = v \frac{d^r u}{dx^r} + r \frac{dv}{dx} \frac{d^{r-1}u}{dx^{r-1}} + \frac{r(r-1)}{1 \,.\, 2} \frac{d^2 v}{dx^2} \frac{d^{r-2}u}{dx^{r-2}} + \&\text{c.}$$

Commer. Epis. Leib. *et* Bern. Vol. I. p. 46, 99.

(13) $uv = x^n (1-x)^n,$

$$\frac{d^r(uv)}{dx^r} = n(n-1)\ldots(n-r+1)(1-x)^n x^{n-r} \left\{1 - \frac{r \,.\, n}{n-r+1} \frac{x}{1-x} \right.$$

$$\left. + \frac{r(r-1)}{1 \,.\, 2} \frac{n(n-1)}{(n-r+1)(n-r+2)} \frac{x^2}{(1-x)^2} - \&\text{c.}\right\}.$$

If $r = n$,

$$\frac{d^n \{x^n(1-x)^n\}}{dx^n} = n(n-1)\ldots 3 \,.\, 2 \,.\, 1 \left\{(1-x)^n - \left(\frac{n}{1}\right)^2 (1-x)^{n-1} x \right.$$

$$\left. + \left\{\frac{n(n-1)}{1 \,.\, 2}\right\}^2 (1-x)^{n-2} x^2 - \&\text{c.}\right\}.$$

Murphy's *Electricity*, p. 7.

(14) $uv = \epsilon^{ax}x^n$,

$$\frac{d^r(uv)}{dx^r} = \epsilon^{ax}\left\{a^r x^n + r.na^{r-1}x^{n-1} + \frac{r(r-1)}{1.2}n(n-1)a^{r-2}x^{n-2} + \&c.\right\}.$$

In the same way, if $uv = \epsilon^{ax}x^r$,

$$\frac{d^n(uv)}{dx^n} = \epsilon^{ax}\left\{a^n x^r + n.ra^{n-1}x^{r-1} + \frac{n(n-1)}{1.2}r(r-1)a^{n-2}x^{r-2} + \&c.\right\}$$

Whence, comparing these expressions, it appears that

$$\left(\frac{d}{dx}\right)^r \epsilon^{ax}x^n = a^{r-n}x^{n-r}\left(\frac{d}{dx}\right)^n \epsilon^{ax}x^r.$$

(15) $uv = x^n \log x$,

$$\frac{d^r(uv)}{dx^r} = n(n-1)\ldots(n-r+1)x^{n-r}\left\{\log x + r.\frac{1}{n-r+1}\right.$$

$$-\frac{r(r-1)}{1.2}\frac{1}{(n-r+1)(n-r+2)}$$

$$\left.+\frac{r(r-1)(r-2)}{1.2.3}\frac{1.2}{(n-r+1)\ldots(n-r+3)} + \&c.\right.$$

If $r = n$,

$$\frac{d^n(x^n\log x)}{dx^n} = n(n-1)\,3.2.1\left\{\log x + \frac{n}{1^2} - \frac{n(n-1)}{(1.2)^2}\right.$$

$$\left.+\frac{n(n-1)(n-2).1.2}{(1.2.3)^2} - \&c.\right\}$$

(16) $uv = \dfrac{(a+x)^m}{(c+x)^n}$,

$$\frac{d^r(uv)}{dx^r} = m(m-1)\ldots(m-r+1)\frac{(a+x)^{m-r}}{(c+x)^n}\left\{1 - \frac{r}{1}.\frac{n}{m-r+1}\frac{a+x}{c+x}\right.$$

$$\left.+\frac{r(r-1)}{1.2}\frac{n(n+1)}{(m-r+1)(m-r+2)}\frac{(a+x)^2}{(c+x)^2} + \&c.\right\}.$$

(17) $uv = \epsilon^{ax} \cos nx \,.\, x^m$,

In this case let $u = \epsilon^{ax} \cos nx$, $v = x^m$.

Then by Ex. (10) if

$$\frac{n}{a} = \tan\phi, \quad \frac{d^p u}{dx^p} = (a^2 + n^2)^{\frac{p}{2}} \epsilon^{ax} \cos(nx + p\phi).$$

Therefore, expanding $\dfrac{d^r(uv)}{dx^r}$ by the Theorem of Leibnitz,

$$\frac{d^r(uv)}{dx^r} = \epsilon^{ax} \; (a^2+n^2)^{\frac{r}{2}} \Big[x^m \cos(nx + r\phi)$$

$$+ r \,.\, m x^{m-1} \frac{\cos\{nx + (r-1)\phi\}}{(a^2+n^2)^{\frac{1}{2}}}$$

$$+ \frac{r(r-1)}{1\,.\,2} m(m-1) x^{m-2} \frac{\cos\{nx + (r-2)\phi\}}{(a^2+n^2)^1} + \&\text{c.}\Big]$$

(18) Let $uv = \epsilon^{ax} X$, X being any function of ~~X~~x.

Then making $u = X$, $v = \epsilon^{ax}$,

$$\frac{d^r(uv)}{dx^r} = \epsilon^{ax} \left\{\frac{d^r X}{dx^r} + r\,.\,a \frac{d^{r-1}X}{dx^{r-1}} + \frac{r(r-1)}{1\,.\,2} a^2 \frac{d^{r-2}X}{dx^{r-2}} + \&\text{c.}\right\}$$

$$= \epsilon^{ax} \left\{\left(\frac{d}{dx}\right)^r + r\,.\,a \left(\frac{d}{dx}\right)^{r-1} + \frac{r(r-1)}{1\,.\,2} a^2 \left(\frac{d}{dx}\right)^{r-2} + \&\text{c.}\right\} X$$

$$= \epsilon^{ax} \left(\frac{d}{dx} + a\right)^r X.$$

Whence it appears that

$$\left(\frac{d}{dx} + a\right)^r X = \epsilon^{-ax} \left(\frac{d}{dx}\right)^r (\epsilon^{ax} X).$$

This result, when generalized, is of great importance in the solution of Differential Equations.

If the function to be differentiated be $(a + bx + cx^2)^n$, the general differential might be found by resolving the trinomial $a + bx + cx^2$ into two factors of the first degree, as into $(x + \alpha)(x + \beta)$, and then differentiating the product $(x + \alpha)^n (x + \beta)^n$ by the Theorem of Leibnitz; but instead of doing so we shall make use of two formulæ given by Lagrange*.

$$\text{Let } u = a + bx + cx^2, \qquad u' = b + 2cx;$$

Then substituting $x + h$ for x in u^n it becomes

$$(u + u'h + ch^2)^n;$$

and $\dfrac{d^r u}{dx^r}$ will be the coefficient of $\dfrac{h^r}{1 . 2 \dots r}$ in the expansion of this trinomial.

Developing it as a binomial, of which $u + u'h$ is the first term, we obtain

$$(u + u'h)^n + n(u + u'h)^{n-1} ch^2 + \frac{n(n-1)}{1 . 2}(u + u'h)^{n-2} c^2 h^4 + \&c.$$

Again, developing each binomial and taking only the terms which multiply h^r, we find that the term in

$$(u + u'h)^n \text{ is } \frac{n(n-1) \dots (n-r+1)}{1 . 2 \dots r} u^{n-r} u'^r;$$

$$\text{in } (u + u'h)^{n-1} h^2 \text{ is } \frac{(n-1) \dots (n-r+2)}{1 . 2 \dots (r-2)} u^{n-r+1} u'^{r-2};$$

$$\text{in } (u + u'h)^{n-2} h^4 \text{ is } \frac{(n-2) \dots (n-r+3)}{1 . 2 \dots r-4} u^{n-r+2} u'^{r-4}; \ \&c.$$

Collecting these terms, and multiplying by $1 . 2 \dots r$, we obtain for the r^{th} differential coefficient of u^n

$$\frac{d^r(u^n)}{dx^r} = n(n-1) \dots (n-r+1) u^{n-r} u'^r \Big\{ 1 + \frac{r(r-1)}{1 . (n-r+1)} \frac{cu}{u'^2}$$
$$+ \frac{r(r-1)(r-2)(r-3)}{1 . 2 (n-r+1)(n-r+2)} \frac{c^2 u^2}{u'^4} + \&c. \Big\} \quad (A)$$

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* *Mémoires de Berlin*, 1772, p. 213.

By developing in a different manner a more convenient formula may be obtained:

$$(u + u'h + ch^2)^n = u^n \left(1 + \frac{u'}{u} h + \frac{c}{u} h^2\right)^n$$

$$= u^n \left\{\left(1 + \frac{u'}{2u} h\right)^2 + \frac{4uc - u'^2}{4u^2} h^2\right\}^n.$$

But $4uc - u'^2 = 4ac - b^2 = e^2$ suppose.

Developing $u^n \left\{\left(1 + \frac{u'}{2u} h\right)^2 + \frac{e^2}{(2u)^2} h^2\right\}^n$ by the binomial theorem, we have

$$u^n \left\{\left(1 + \frac{u'}{2u} h\right)^{2n} + n\left(1 + \frac{u'}{2u} h\right)^{2n-2} \frac{e^2}{(2u)^2} h^2\right.$$

$$\left. + \frac{n(n-1)}{1.2}\left(1 + \frac{u'}{2u} h\right)^{2n-4} \frac{e^4}{(2u)^4} h^4 + \&c.\right\},$$

and the r^{th} differential of u^n is the coefficient of h^r in this expansion multiplied by $1.2...r$. Now expanding each term by the binomial theorem, we have for the coefficient of

h^r in the first term $\left(\frac{u'}{2}\right)^r \frac{1}{u^r} \frac{2n(2n-1)...(2n-r+1)}{1.2...r}$,

... second $\left(\frac{u'}{2}\right)^{r-2} \frac{1}{2^2 u^r} \frac{(2n-2)...(2n-r+1)}{1.2...(r-2)} \frac{n}{1} e^2$,

... third $\left(\frac{u'}{2}\right)^{r-4} \frac{1}{2^4 u^r} \frac{(2n-4)...(2n-r+1)}{1.2...(r-4)} \frac{n(n-1)}{1.2} e^4$,

and so on. Collecting these terms and multiplying by $1.2...r$, we find

$$\frac{d^r(u^n)}{dx^r} = 2n(2n-1)...(2n-r+1)\left(\frac{u'}{2}\right)^r u^{n-r}\left\{1 + \frac{n}{1}\frac{r(r-1)}{2n(2n-1)}\frac{e^2}{u'^2}\right.$$

$$\left. + \frac{n(n-1)}{1.2}\frac{r(r-1)(r-2)(r-3)}{2n(2n-1)...(2n-3)}\frac{e^4}{u'^4} + \&c.\right\}...(B).$$

(19) Let $u^{\cancel{n}} = (a^2 + x^2)^n$.

Here $u' = 2x$, $e = 4a^2$, and if we make $r = n$, we find by formula (B),

$$\frac{d^n(a^2+x^2)^n}{dx^n} = 2n(2n-1)\ldots(n+1)x^n\left\{1 + \frac{n^2}{1}\cdot\frac{n-1}{2n(2n-1)}\,\frac{a^2}{x^2}\right.$$

$$\left. + \frac{\{n(n-1)\}^2}{1\,.\,2}\;\frac{(n-2)\,(n-3)}{2n\ldots(2n-3)}\,\frac{a^4}{x^4} + \&\text{c.}\right\}.$$

(20) Let $u^n = \dfrac{1}{a^2 + x^2}$.

The r^{th} differential of this function may be found as in the last example, but the following method gives it under a form which is more convenient in practice;

$$\frac{1}{a^2 + x^2} = -\frac{1}{2a\,(-)^{\frac{1}{2}}}\left\{\frac{1}{x + a\,(-)^{\frac{1}{2}}} - \frac{1}{x - a\,(-)^{\frac{1}{2}}}\right\}.$$

Differentiating r times,

$$\left(\frac{d}{dx}\right)^r\cdot\frac{1}{a^2 + x^2}$$

$$= (-)^{r+1}\cdot\frac{r(r-1)\ldots2\,.\,1}{2a\,(-)^{\frac{1}{2}}}\left\{\frac{1}{\{x+a(-)^{\frac{1}{2}}\}^{r+1}} - \frac{1}{\{x-a(-)^{\frac{1}{2}}\}^{r+1}}\right\}$$

$$= (-)^{r+1}\frac{r(r-1)\ldots2\,.\,1}{2a\,(-)^{\frac{1}{2}}}\left\{\frac{\{x-a(-)^{\frac{1}{2}}\}^{r+1} - \{x+a(-)^{\frac{1}{2}}\}^{r+1}}{(a^2 + x^2)^{r+1}}\right\}.$$

Now let $\theta = \tan^{-1}\dfrac{a}{x}$, so that

$$x = (a^2 + x^2)^{\frac{1}{2}}\cos\theta, \quad a = (a^2 + x^2)^{\frac{1}{2}}\sin\theta,$$

and therefore

$$\{x - a\,(-)^{\frac{1}{2}}\}^{r+1} = (a^2 + x^2)^{\frac{r+1}{2}}\{\cos\,(r+1)\,\theta - (-)^{\frac{1}{2}}\sin\,(r+1)\,\theta\},$$

$$\{x + a\,(-)^{\frac{1}{2}}\}^{r+1} = (a^2 + x^2)^{\frac{r+1}{2}}\{\cos\,(r+1)\,\theta + (-)^{\frac{1}{2}}\sin\,(r+1)\,\theta\}.$$

Hence we have

$$\left(\frac{d}{dx}\right)^r \cdot \frac{1}{a^2+x^2} = \frac{(-)^r r(r-1)\ldots 2.1}{a} \frac{\sin(r+1)\theta}{(a^2+x^2)^{\frac{r+1}{2}}}.$$

Liouville, *Jour: de l'Ecole Polytechnique*, Cah. 21, p. 157.

(21) In the same way if we had the function

$$u = \frac{x}{a^2+x^2},$$

we should find

$$\left(\frac{d}{dx}\right)^r \frac{x}{a^2+x^2} = (-)^r r(r-1)\ldots 2.1 \frac{\cos(r+1)\theta}{(a^2+x^2)^{\frac{r+1}{2}}}.$$

Liouville, *Ib.* p. 156.

These results are useful in the theory of definite integrals.

In the following examples the functions are reduced to the required forms by differentiation in the same way as in Ex. 11.

(22) Let $u = \dfrac{x}{(1-x^2)^{\frac{1}{2}}}$; $\dfrac{du}{dx} = \dfrac{1}{(1-x^2)^{\frac{3}{2}}}$.

Therefore $\dfrac{d^r}{dx^r}\dfrac{x}{(1-x^2)^{\frac{1}{2}}} = \dfrac{d^{r-1}}{dx^{r-1}}\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$,

and by formula (B),

$$\frac{d^r u}{dx^r} = \frac{3.4\ldots(r+1)x^{r-1}}{(1-x^2)^{r+\frac{1}{2}}}\left\{1 + \frac{3}{2}\frac{(r-1)(r-2)}{3.4}\frac{1}{x^2}\right.$$
$$\left. + \frac{3.5}{2.4}\frac{(r-1)(r-2)(r-3)(r-4)}{3.4.5.6}\frac{1}{x^4} + \&c.\right.$$

(23) $u = \sin^{-1}\dfrac{x}{a}$; $\dfrac{du}{dx} = \dfrac{1}{(a^2-x^2)^{\frac{1}{2}}}$,

$$\frac{d^r u}{dx^r} = \frac{d^{r-1}}{dx^{r-1}} \frac{1}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= \frac{1.2\ldots(r-1)\,x^{r-1}}{(a^2 - x^2)^{r-\frac{1}{2}}} \Big\{1 + \frac{1}{2}\,\frac{(r-1)(r-2)}{1.2}\,\frac{a^2}{x^2}$$

$$+ \frac{1.3}{2.4}\,\frac{(r-1)\ldots(r-4)}{1.2.3.4}\,\frac{a^4}{x^4} + \&c.\Big\} \text{ by (B).}$$

(24) Let $u = \tan^{-1}\dfrac{x}{a}$; $\dfrac{du}{dx} = \dfrac{a}{a^2 + x^2}$

$$\frac{d^r u}{dx^r} = a \left(\frac{d}{dx}\right)^{r-1} \frac{1}{a^2 + x^2}$$

$$= (-)^{r-1}(r-1)(r-2)\ldots 2.1\,\frac{\sin r\theta}{(a^2 + x^2)^{\frac{r}{2}}} \text{ by Ex. 20.}$$

where $\theta = \tan^{-1}\dfrac{a}{x} = \dfrac{\pi}{2} - \tan^{-1}\dfrac{x}{a}$.

The method employed by Lagrange may be used for the determination of the successive differentials of other functions.

(25) Let $u = \epsilon^{cx^2}$.

If x become $x + h$, u becomes $\epsilon^{c(x+h)^2} = \epsilon^{c(x^2 + 2xh + h^2)} = \epsilon^{cx^2}.\epsilon^{2cxh}.\epsilon^{ch^2}$.

$$\text{Now } \epsilon^{2cxh} = 1 + 2cxh + \frac{(2cx)^2}{1.2}h^2 + \frac{(2cx)^3}{1.2.3}h^3 + \&c.$$

$$\text{and } \epsilon^{ch^2} = 1 + ch^2 + \frac{c^2}{1.2}h^4 + \frac{c^3}{1.2.3}h^6 + \&c.$$

Multiplying these together, taking only the coefficient of h^r, and multiplying it by $1.2\ldots r$, we find

$$\frac{d^r u}{dx^r} = \epsilon^{cx^2}\Big\{c^r(2x)^r + r(r-1)\,c^{r-1}(2x)^{r-2}$$

$$+ \frac{r(r-1)\ldots(r-3)}{1.2}\,c^{r-2}(2x)^{r-4} + \&c.\Big\}$$

(26) From this we can determine the successive differentials of $\cos x^2$ and $\sin x^2$.

Let $u = \cos x^2 + (-)^{\frac{1}{2}} \sin x^2 = \epsilon^{(-)^{\frac{1}{2}} x^2}$.

Then differentiating by the preceding formula

$$\frac{d^r u}{dx^r} = \epsilon^{(-)^{\frac{1}{2}} x^2} \Big\{ (-)^{\frac{r}{2}} (2x)^r + (-)^{\frac{r-1}{2}} r(r-1)(2x)^{r-2}$$

$$+ (-)^{\frac{r-2}{2}} \frac{r(r-1)\dots(r-3)}{1.2} (2x)^{r-4} + \&c. \Big\}.$$

Now generally $(-)^{\frac{p}{2}} = \epsilon^{(-)^{\frac{1}{2}} p \frac{\pi}{2}}$,

and $\epsilon^{(-)^{\frac{1}{2}} x^2} \epsilon^{(-)^{\frac{1}{2}} p \frac{\pi}{2}} = \cos\left(x^2 + p\dfrac{\pi}{2}\right) + (-)^{\frac{1}{2}} \sin\left(x^2 + p\dfrac{\pi}{2}\right)$.

Therefore making these substitutions, and as

$$\frac{d^r u}{dx^r} = \left(\frac{d}{dx}\right)^r \cos x^2 + (-)^{\frac{1}{2}} \left(\frac{d}{dx}\right)^r \sin x^2,$$

equating possible and impossible parts, we have

$$\frac{d^r(\cos x^2)}{dx^r} = (2x)^r \cos\left(x^2 + r\frac{\pi}{2}\right) + r(r-1)(2x)^{r-2} \cos\left\{x^2 + (r-1)\frac{\pi}{2}\right\}$$

$$+ \frac{r(r-1)\dots(r-3)}{1.2} (2x)^{r-4} \cos\left\{x^2 + (r-2)\frac{\pi}{2}\right\} + \&c.\,;$$

and

$$\frac{d^r(\sin x^2)}{dx^r} = (2x)^r \sin\left(x^2 + r\frac{\pi}{2}\right) + r(r-1)(2x)^{r-2} \sin\left\{x^2 + (r-1)\frac{\pi}{2}\right\}$$

$$+ \frac{r(r-1)\dots(r-3)}{1.2} (2x)^{r-4} \sin\left\{x^2 + (r-2)\frac{\pi}{2}\right\} + \&c.$$

(27) Let $u = \dfrac{1}{\epsilon^x + 1}$.

We might in this case expand the function and differentiate r times each term in the development, but as this would give $\frac{d^r u}{dx^r}$ expressed in an infinite series, the following method, due to Laplace*, is to be preferred. It is easily seen on effecting two or three differentiations that the form of $\frac{d^r u}{dx^r}$ must be

$$\frac{a_r \epsilon^{rx} + a_{r-1}\epsilon^{(r-1)x} + a_{r-2}\epsilon^{(r-2)x} + \&c. + a_1 \epsilon^x}{(\epsilon^x + 1)^{r+1}}.$$

Hence multiplying by $(\epsilon^x + 1)^{r+1}$ we must have

$$(\epsilon^x + 1)^{r+1}\frac{d^r u}{dx^r} = a_r \epsilon^{rx} + a_{r-1}\epsilon^{(r-1)x} + \&c. + a_1 \epsilon^x \quad (1).$$

Now as $u = \epsilon^{-x} - \epsilon^{-2x} + \epsilon^{-3x} - \&c.$

$$\frac{d^r u}{dx^r} = (-)^r \{1^r \epsilon^{-x} - 2^r \epsilon^{-2x} + 3^r \epsilon^{-3x} - 4^r \epsilon^{-4x} + \&c.\} \quad (2).$$

Also, developing $(\epsilon^x + 1)^{r+1}$ we have

$$(\epsilon^x + 1)^{r+1} = \epsilon^{(r+1)x} + \frac{(r+1)}{1}\epsilon^{rx} + \frac{(r+1)\,r}{1\,.\,2}\epsilon^{(r-1)x}$$
$$+ \frac{(r+1)\,r\,(r-1)}{1\,.\,2\,.\,3}\epsilon^{(r-2)x} + \&c. \quad (3).$$

The product of (2) and (3) must be equal to the second side of (1), and as this last consists of a finite number of terms having positive indices, the terms in the product of (2) and (3) which contain negative indices must disappear of themselves. Hence taking the terms with positive indices only

$$(\epsilon^x + 1)^{r+1}\frac{d^r u}{dx^r} = (-)^r \left[1^r \epsilon^{rx} - \left\{2^r - \frac{(r+1)}{1}1^r\right\}\epsilon^{r-1x}\right.$$
$$\left. + \left\{3^r - \frac{(r+1)}{1}2^r + \frac{(r+1)\,r}{1.2}1^r\right\}\epsilon^{(r-2)x} + \&c.\right]$$

and therefore

$$\frac{d^r u}{dx^r} = \frac{(-)^r \left[1^r \epsilon^{rx} - \left\{2^r - \frac{(r+1)}{1}1^r\right\}\epsilon^{(r-1)x} + \left\{3^r - \frac{(r+1)}{1}2^r + \frac{(r+1)\,r}{1\,.\,2}1^r\right\}\epsilon^{(r-2)x} + \&c.\right]}{(\epsilon^x + 1)^{r+1}}$$

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* *Mémoires de l'Académie*, 1777, p. 108.

SEC. 2. *Functions of two or more Variables.*

If u be a function of two variables x and y,

$$\frac{d^{r+s}u}{dy^s\,dx^r} = \frac{d^{r+s}u}{dx^r\,dy^s}.$$

Ex. (1) $u = x^m y^n;\quad r = 1,\quad s = 1,$

$$\frac{du}{dx} = mx^{m-1}y^n;\quad \frac{du}{dy} = nx^m y^{n-1};$$

$$\frac{d^2u}{dy\,dx} = mnx^{m-1}y^{n-1} = \frac{d^2u}{dx\,dy}.$$

(2) $u = \dfrac{x^2+y^2}{x^2-y^2};\quad r = 1,\quad s = 1,$

$$\frac{d^2u}{dy\,dx} = -8xy\frac{x^2+y^2}{(x^2-y^2)^3} = \frac{d^2u}{dx\,dy}.$$

(3) $u = y^x;\quad r = 1,\quad s = 1,$

$$\frac{du}{dx} = y^x\log y,\quad \frac{du}{dy} = xy^{x-1};$$

$$\frac{d^2u}{dy\,dx} = y^{x-1}(1 + x\log y) = \frac{d^2u}{dx\,dy}.$$

(4) $u = \sin(mx + ny);$

$$\frac{d^ru}{dx^r} = m^r\sin\left(mx + ny + r\frac{\pi}{2}\right),$$

$$\frac{d^su}{dy^s} = n^s\sin\left(mx + ny + s\frac{\pi}{2}\right),$$

$$\frac{d^{r+s}u}{dy^s\,dx^r} = m^rn^s\sin\left\{mx + ny + (r+s)\frac{\pi}{2}\right\} = \frac{d^{r+s}u}{dx^r\,dy^s}.$$

(5) $u = \sin\dfrac{x}{y};\quad r = 2,\quad s = 1,$

$$\frac{d^3u}{dy\,dx^2} = \frac{2}{y^2}\sin\frac{x}{y} + \frac{x}{y^4}\cos\frac{x}{y} = \frac{d^3u}{dx^2\,dy}.$$

(6) $$u = \sin^{-1}\frac{x}{y}; \quad r = 1, \quad s = 1,$$

$$\frac{d^2u}{dy\,dx} = -\frac{y}{(y^2 - x^2)^{\frac{3}{2}}} = \frac{d^2u}{dx\,dy}.$$

(7) $$u = \tan^{-1}\frac{x}{y}; \quad r = 1, \quad s = 1;$$

$$\frac{d^2u}{dy\,dx} = \frac{x^2 - y^2}{(y^2 + x^2)^2} = \frac{d^2u}{dx\,dy}.$$

(8) $$u = x\sin y + y\sin x; \quad r = 1, \quad s = 1;$$

$$\frac{d^2u}{dy\,dx} = \cos y + \cos x = \frac{d^2u}{dx\,dy}.$$

(9) $$u = \sin x\cos y; \quad r = 2, \quad s = 2;$$

$$\frac{d^4u}{dy^2\,dx^2} = \sin x\cos y = \frac{d^4u}{dx^2\,dy^2} = \frac{d^4u}{dx\,dy\,dx\,dy}.$$

Generally, in a function of any number of variables, the order of differentiation is indifferent.

(10) $$u = \frac{x^2y}{a^2 - z^2};$$

$$\frac{du}{dx} = \frac{2xy}{a^2 - z^2}, \quad \frac{du}{dy} = \frac{x^2}{a^2 - z^2}, \quad \frac{du}{dz} = \frac{2x^2yz}{(a^2 - z^2)^2},$$

$$\frac{d^2u}{dx\,dy} = \frac{2x}{a^2 - z^2} = \frac{d^2u}{dy\,dx},$$

$$\frac{d^2u}{dx\,dz} = \frac{4xyz}{(a^2 - z^2)^2} = \frac{d^2u}{dz\,dx}.$$

$$\frac{d^2u}{dy\,dz} = \frac{2x^2z}{(a^2-z^2)^2} = \frac{d^2u}{dz\,dy},$$

$$\frac{d^3u}{dx\,dy\,dz} = \frac{4xz}{(a^2-z^2)^2} = \frac{d^3u}{dy\,dx\,dz} = \frac{d^3u}{dz\,dy\,dx},$$

$$= \frac{d^3u}{dz\,dx\,dy} = \frac{d^3u}{dy\,dz\,dx} = \frac{d^3u}{dx\,dz\,dy}.$$

(11) $$u = \frac{\epsilon^z y}{(x^2+y^2)^{\frac{1}{2}}},$$

$$\frac{d^2u}{dx\,dy} = \frac{\epsilon^z x\,(2y^2-x^2)}{(x^2+y^2)^{\frac{5}{2}}} = \frac{d^2u}{dy\,dx},$$

$$\frac{d^2u}{dx\,dz} = -\frac{\epsilon^z xy}{(x^2+y^2)^{\frac{3}{2}}} = \frac{d^2u}{dz\,dx},$$

$$\frac{d^2u}{dy\,dz} = \frac{\epsilon^z x^2}{(x^2+y^2)^{\frac{3}{2}}} = \frac{d^2u}{dz\,dy}.$$

(12) $$u = \frac{xy}{ax+bz},$$

$$\frac{d^3u}{dz^2dy} = \frac{2b^2x}{(ax+bz)^3} = \frac{d^3u}{dy\,dz^2} = \frac{d^3u}{dz\,dy\,dz},$$

$$\frac{d^3u}{dx\,dz^2} = \frac{2b^2y\,(bz-2ax)}{(ax+bz)^4} = \frac{d^3u}{dz^2dx} = \frac{d^3u}{dz\,dx\,dz}.$$

The general total differential of two variables is given in terms of the general partial differentials by the formula,

$$d^nu = \frac{d^nu}{dx^n}dx^n + n\frac{d^nu}{dx^{n-1}dy}dx^{n-1}dy$$

$$+ \frac{n(n-1)}{1\,.\,2}\,\frac{d^nu}{dx^{n-2}dy^2}dx^{n-2}dy^2 + \&c.,$$

the law of the coefficients being that of Newton's Binomial Theorem.

(13) $u = x^m y^n$;

$$d^4u = m(m-1)\ldots(m-3)\left\{x^{m-4}y^n dx^4 + 4\,\frac{n}{m-3}\,x^{m-3}y^{n-1}dx^3dy\right.$$

$$+\,6\,\frac{n(n-1)}{(m-2)\,(m-3)}\,x^{m-2}y^{n-2}dx^2dy^2$$

$$+\,4\,\frac{n\,(n-1)\,(n-2)}{(m-1)\,(m-2)\,(m-3)}\,x^{m-1}y^{n-3}dx\,dy^3$$

$$\left.+\,\frac{n\,(n-1)\,(n-2)\,(n-3)}{m\,(m-1)\,(m-2)\,(m-3)}\,x^m y^{n-4}dy^4\right\}.$$

(14) $u = \epsilon^{ax+by}$;

$$d^3u = (a^3dx^3 + 3a^2b\,dx^2dy + 3ab^2dx\,dy^2 + b^3dy^3)\,\epsilon^{ax+by}.$$

(15) $u = \sin mx \sin ny$;

$$d^4u = (m^4dx^4 + 6m^2n^2dx^2dy^2 + n^4dy^4)\sin mx \sin ny$$

$$-\,4mn\,(m^2dx^3dy + n^2dx\,dy^3)\cos mx \cos ny.$$

(16) $u = \log(ax + by)$;

$$d^2u = -\,(a^2dx^2 + 2ab\,dx\,dy + b^2dy^2)\,\frac{1}{(ax+by)^2}.$$

(17) $u = (x^2 + y^2)^{\frac{1}{2}}$;

$$d^2u = (y^2dx^2 - 2xy\,dx\,dy + x^2dy^2)\,\frac{1}{(x^2+y^2)^{\frac{3}{2}}}.$$

(18) $u = \sin^{-1}\dfrac{x}{y}$;

$$d^2u = \left\{x\,dx^2 - 2y\,dx\,dy + x\,\frac{(2y^2 - x^2)}{y^2}\,dy^2\right\}\frac{1}{(y^2-x^2)^{\frac{3}{2}}}.$$

There is a very important theorem (due to Euler) regarding homogeneous functions of any number of variables, which from the frequent applications made of it ought to be noticed in this place.

If u be a homogeneous algebraic function of n dimensions of r variables $x, y, z, \ldots$; then

$$x\frac{du}{dx} + y\frac{du}{dy} + z\frac{du}{dz} \ldots = nu.$$

From this may be derived a series of equations of the form

$$x^m\frac{d^m u}{dx^m} + y^m\frac{d^m u}{dy^m} + z^m\frac{d^m u}{dz^m} + \&\text{c.}$$

$$+ 1 \,.\, 2 \ldots m\Sigma\,\frac{\left(\frac{d}{dx}\right)^\alpha \left(\frac{d}{dy}\right)^\beta \left(\frac{d}{dz}\right)^\gamma \ldots u}{1.2\ldots\alpha.1.2\ldots\beta.1.2\ldots\gamma\ldots},$$

$$= n(n-1)\ldots(n-m+1)\,u,$$

where $\alpha + \beta + \gamma + \ldots = m$. Euler, *Calc. Diff.* p. 188.

In applying this theorem to transcendental functions of algebraical functions, it is to be observed that it is not sufficient that these last should be homogeneous, it is also necessary that they should be of zero dimensions, as, otherwise, in the development of the transcendental function the degree of each term would be different, and the function when expanded not homogeneous.

(19) Let $u = \dfrac{y^3 + x^4}{y - x}$. Then $n = 2$ and

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{2y^4 - 2y^3x + 2yx^3 - 2x^4}{(y-x)^2} = \frac{2(y^3 + x^3)}{y - x}.$$

(20) $u = \dfrac{x^{\frac{1}{2}} + y^{\frac{1}{2}}}{x + y}$. Then $n = -\frac{1}{2}$, and

$$x\frac{du}{dx} + y\frac{du}{dy} = -\frac{1}{2}\,\frac{(yx^{\frac{1}{2}} + xy^{\frac{1}{2}} + x^{\frac{3}{2}} + y^{\frac{3}{2}})}{(x+y)^2} = -\frac{1}{2}\,\frac{(x^{\frac{1}{2}} + y^{\frac{1}{2}})}{(x+y)}.$$

(21) $u = \sin^{-1}\left(\dfrac{x - y}{x + y}\right)^{\frac{1}{2}}$. Then $n = 0$, and

$$x\frac{du}{dx} + y\frac{du}{dy} = \frac{yx - xy}{(x+y)\{2y(x-y)\}^{\frac{1}{2}}} = 0.$$

(22) $u = (x^2 + y^2)^{\frac{1}{2}}, \quad n = 1,$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx\,dy} + y^2 \frac{d^2 u}{dy^2} = \frac{x^2 y^2 - 2x^2 y^2 + x^2 y^2}{(x^2 + y^2)^{\frac{3}{2}}} = 0.$$

(23) $u = x(2xy + y^2)^{\frac{1}{2}}, \quad n = 2,$

$$x^2 \frac{d^2 u}{dx^2} + 2xy \frac{d^2 u}{dx\,dy} + y^2 \frac{d^2 u}{dy^2}$$

$$= \frac{(3xy^2 + 2y^3)\,x^2 + 2xy\,(3x^2 y + 3xy^2 + y^3) - x^3 y^2}{(2xy + y^2)^{\frac{3}{2}}}$$

$$= 2x\,(2xy + y^2)^{\frac{1}{2}}.$$

(24) If u be a homogeneous and symmetrical function of x and y of n dimensions, so that

$$u = x^n f\left(\frac{y}{x}\right) = y^n f\left(\frac{x}{y}\right);$$

and if it be expanded in terms of x so as to be of the form

$$\Sigma\,.\,(Q_i x^i y^{n-i}),$$

then will $\quad \Sigma\,\{(2i - n)\,Q_i\} = 0.$

As u is homogeneous of n dimensions, we have

$$nu = x\frac{du}{dx} + y\frac{du}{dy},$$

and as it is symmetrical in x and y, we have

$$x\frac{du}{dx} = y\frac{du}{dy} \text{ when } x = y, \text{ so that}$$

$$2x\frac{du}{dx} - nu = 0 \text{ when } x = y.$$

Substituting the expansion of u in this equation, we get

$$\Sigma\,\{(2i - n)\,Q_i x^n\} = 0, \text{ or}$$

$$\Sigma\,\{(2i - n)\,Q_i\} = 0.$$[*]

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[*] This extension of a property of Laplace's Functions was communicated to me by Mr Archibald Smith.

CHAPTER III.

CHANGE OF THE INDEPENDENT VARIABLE.

SECT. I. *Functions of One Variable.*

IF $y=f(x)$ and therefore $x=f^{-1}(y)$, the successive differential coefficients of y with respect to x are transformed into those of x with respect to y by means of the formulæ,

$$\frac{dy}{dx}=\frac{1}{\dfrac{dx}{dy}},\quad \frac{d^2y}{dx^2}=-\frac{\dfrac{d^2x}{dy^2}}{\left(\dfrac{dx}{dy}\right)^3},$$

$$\frac{d^3y}{dx^3}=\frac{3\left(\dfrac{d^2x}{dy^2}\right)^2-\dfrac{dx}{dy}\dfrac{d^3x}{dy^3}}{\left(\dfrac{dx}{dy}\right)^5},$$

and similarly for higher orders. The reader will find the demonstration of a general formula for the change of the n^{th} differential coefficient in a Memoir by Mr Murphy, in the *Philosophical Transactions*, 1837, p. 210. The expression is of necessity extremely complicated, and the demonstration would not be intelligible without so much preliminary matter that I cannot insert it here, and I must therefore content myself with referring the reader to the original Memoir.

If $u=f(y)$ and $y=\phi(x)$ so that u may also be considered as a function of x, the successive differential coefficients of u with respect to y may be transformed into those of u with respect to x by the formulæ

$$\frac{du}{dy} = \frac{\frac{du}{dx}}{\frac{dy}{dx}}, \quad \frac{d^2u}{dy^2} = \frac{\frac{d^2u}{dx^2}\frac{dy}{dx} - \frac{d^2y}{dx^2}\frac{du}{dx}}{\left(\frac{dy}{dx}\right)^3},$$

$$\frac{d^3u}{dy^3} = \frac{\frac{dy}{dx}\left(\frac{d^3u}{dx^3}\frac{dy}{dx} - \frac{d^3y}{dx^3}\frac{du}{dx}\right) - 3\frac{d^2y}{dx^2}\left(\frac{d^2u}{dx^2}\frac{dy}{dx} - \frac{d^2y}{dx^2}\frac{du}{dx}\right)}{\left(\frac{dy}{dx}\right)^5}.$$

The general formula for this transformation will be found in the Memoir of Mr Murphy before referred to, but the result is of such extreme complexity, that it happens fortunately that we have seldom to employ these transformations for high orders of differentials; and where this is necessary, that the nature of the case usually gives us the means of simplification.

(1) Change the formula

$$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx} + (y - a)\frac{d^2y}{dx^2} = 0,$$

into one where y is the independent variable.

The result is

$$1 + \left(\frac{dx}{dy}\right)^2 - (y - a)\frac{d^2x}{dy^2} = 0.$$

(2) The expression for the radius of curvature when x is the independent variable is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}.$$

When y is made the independent variable, it becomes

$$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}.$$

(3) Transform

$$\frac{d^2y}{dx^2} - x\left(\frac{dy}{dx}\right)^3 + \epsilon^y\left(\frac{dy}{dx}\right)^3 = 0,$$

into an equation in which y is the independent variable.

The result is

$$\frac{d^2x}{dy^2} + x - \epsilon^y + 0.$$

(4) Change the variable in

$$\frac{du}{dy} + \frac{u}{(1+y^2)^{\frac{1}{2}}} = a$$

from y to x, when $x = \log\{y + (1 + y^2)^{\frac{1}{2}}\}$.

The result is $\dfrac{du}{dx} + u = \dfrac{a}{2}(\epsilon^x + \epsilon^{-x})$.

(5) Change the variable in

$$y^2\frac{d^2u}{dy^2} + Ay\frac{du}{dy} + Bu = 0$$

from y to x when $y = \epsilon^x$. The result is

$$\frac{d^2u}{dx^2} + (A - 1)\frac{du}{dx} + Bu = 0.$$

(6) There is a very convenient formula by which we can change generally the independent variable in $y^n\dfrac{d^nu}{dy^n}$ from y to x when $y = \epsilon^x$. Taking the symbol of operation alone,

$$y^n\left(\frac{d}{dy}\right)^n = \epsilon^{nx}\left(\epsilon^{-x}\frac{d}{dx}\right)^n$$

$$= \epsilon^{nx}\left(\epsilon^{-x}\frac{d}{dx}\right)\left(\epsilon^{-x}\frac{d}{dx}\right)\left(\epsilon^{-x}\frac{d}{dx}\right) \text{...... to } n \text{ factors.}$$

This may be put under the form

$$\left\{\epsilon^{(n-1)x}\frac{d}{dx}\epsilon^{-(n-1)x}\right\}\left\{\epsilon^{(n-2)x}\frac{d}{dx}\epsilon^{-(n-2)x}\right\}\text{......}\left(\epsilon^x\frac{d}{dx}\epsilon^{-x}\right)\frac{d}{dx}.$$

Now by the theorem given in Ex. 18, of Chap. II. Sec. 1. we have generally

$$\left(\frac{d}{dx} - a\right) = \epsilon^{ax} \cdot \frac{d}{dx} \cdot \epsilon^{-ax}.$$

Hence, substituting these binomial factors, we find

$$y^n \frac{d^n u}{dy^n} = \left[\left\{\frac{d}{dx} - (n-1)\right\}\left\{\frac{d}{dx} - (n-2)\right\} \dots \left(\frac{d}{dx} - 1\right) \frac{d}{dx}\right] u.$$

(7) Change the independent variable in

$$(1 - y^2) \frac{d^2 u}{dy^2} - y \frac{du}{dy} + n^2 u = 0,$$

from y to x, having given $y = \cos x$. The result is

$$\frac{d^2 u}{dx^2} + n^2 u = 0.$$

(8) Change the independent variable in

$$(1 - y^2)^2 \frac{d^2 u}{dy^2} - 2y (1 - y^2) \frac{du}{dy} + \frac{2a}{1 - y} u = 0,$$

from y to x, having given $y = \frac{\epsilon^{2x} - 1}{\epsilon^{2x} + 1}$. The result is

$$\frac{d^2 u}{dx^2} + a (\epsilon^{2x} + 1) u = 0.$$

(9) Change the variable in

$$(a + y)^3 \frac{d^3 u}{dy^3} + 3 (a + y)^2 \frac{d^2 u}{dy^2} + (a + y) \frac{du}{dy} + bu = 0,$$

from y to x, having given $x = \log (a + y)$. The result is

$$\frac{d^3 u}{dx^3} + bu = 0.$$

(10) Transform

$$u + \frac{1}{x} \frac{du}{dx} + \frac{d^2 u}{dx^2} = 0,$$

from x to θ, having given $x^2 = 4\theta$. The resulting equation is

$$u + \frac{du}{d\theta} + \theta \frac{d^2 u}{d\theta^2} = 0.$$

Fourier, *Traité de la Chaleur*, p. 376.

(11) Transform $$\frac{\dfrac{d^2 y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

into a function where s is the independent variable, having given that

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2;$$

the result is $$\frac{d^2 y}{ds^2}\frac{dx}{ds} - \frac{d^2 x}{ds^2}\frac{dy}{ds}.$$

(12) Transform $$p = \frac{x\dfrac{dy}{dx} - y}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}$$

into a function of r and θ, having given $x = r\cos\theta$, $y = r\sin\theta$. In this case we consider r to be a function of θ; differentiating therefore x and y on this hypothesis,

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta;$$

and therefore $$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Substituting this expression for $\dfrac{dy}{dx}$, we find

$$p = \frac{r^2}{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}}.$$

(13) Transform $\rho = \dfrac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}$

into a function where θ is the independent variable, having given

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Proceeding as in the previous example we find

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

(14) Express $t = \dfrac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}$

in r and θ, having given

$$x = r\cos\theta, \quad y = r\sin\theta;$$

the result is $t = r\dfrac{d\theta}{dr}$.

SEC. 2. *Functions of two or more variables.*

Let u be a function of two variables, x and y, so that

$$u = f(x, y);$$

then to express $\dfrac{du}{dx}$ and $\dfrac{du}{dy}$ in terms of two new variables

r and θ, of which x and y are functions given by the equations

$$x = \phi(r, \theta), \qquad y = \psi(r, \theta),$$

we proceed as follows. We have

$$\frac{du}{dr} = \frac{du}{dx} \cdot \frac{dx}{dr} + \frac{du}{dy} \cdot \frac{dy}{dr},$$

$$\frac{du}{d\theta} = \frac{du}{dx} \cdot \frac{dx}{d\theta} + \frac{du}{dy} \cdot \frac{dy}{d\theta}.$$

Eliminating $\dfrac{du}{dy}$ we find

$$\frac{du}{dx} = \frac{\dfrac{du}{dr} \cdot \dfrac{dy}{d\theta} - \dfrac{du}{d\theta} \cdot \dfrac{dy}{dr}}{\dfrac{dx}{dr} \cdot \dfrac{dy}{d\theta} - \dfrac{dy}{dr} \cdot \dfrac{dx}{d\theta}}.$$

Eliminating $\dfrac{du}{dx}$ we find

$$\frac{du}{dy} = -\frac{\dfrac{du}{dr} \cdot \dfrac{dx}{d\theta} - \dfrac{du}{d\theta} \cdot \dfrac{dx}{dr}}{\dfrac{dx}{dr} \cdot \dfrac{dy}{d\theta} - \dfrac{dy}{dr} \cdot \dfrac{dx}{d\theta}}.$$

If r and θ be given explicitly in terms of x and y, we have at once

$$\frac{du}{dx} = \frac{du}{dr} \cdot \frac{dr}{dx} + \frac{du}{d\theta} \cdot \frac{d\theta}{dx},$$

$$\frac{du}{dy} = \frac{du}{dr} \cdot \frac{dr}{dy} + \frac{du}{d\theta} \cdot \frac{d\theta}{dy}.$$

For the successive differentials we proceed in the same manner; and if there be more than two independent variables, the only difference is that the expressions become more complicated. Such cases however seldom occur.

If the independent variables enter into multiple integrals, we cannot substitute directly the values of the original differentials in terms of the new variables, because one is supposed to vary while the others are constant. To introduce this condition we proceed as follows. Let for example there be a double integral $\iint V\,dx\,dy$, and let

$$x = \phi(r, \theta), \quad y = \psi(r, \theta),$$

so that
$$dx = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta,$$

$$dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta.$$

Since x is to vary when y is constant and *vice versâ*, we must make $dy = 0$ when we wish to find dx, and $dx = 0$ when we wish to find dy. Taking the latter condition, we have the two simultaneous equations

$$0 = \frac{dx}{dr}dr + \frac{dx}{d\theta}d\theta,$$

$$dy = \frac{dy}{dr}dr + \frac{dy}{d\theta}d\theta.$$

Eliminating $d\theta$ between these we find

$$\frac{dx}{d\theta}dy = \left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right)dr.$$

From this it follows that when $dy = 0$, $dr = 0$. Hence we have

$$dx = \frac{dx}{d\theta}d\theta.$$

Substituting these values in the double integral it becomes

$$\iint V\left(\frac{dx}{d\theta}\frac{dy}{dr} - \frac{dx}{dr}\frac{dy}{d\theta}\right)dr\,d\theta.$$

If we had three variables x, y, z to be transformed into three others p, q, r, we should have three equations of the form

$$dx = Pdp + Qdq + Rdr,$$
$$dy = P_1dp + Q_1dq + R_1dr,$$
$$dz = P_2dp + Q_2dq + R_2dr;$$

and we should determine dx by supposing $dy = 0$, and $dz = 0$, and then eliminating two of the three quantities dp, dq, dr. Supposing we eliminate the last two we have $dx = Mdp$, M being a function of p, q, r. From this it follows that when $dx = 0$, $dp = 0$. Hence supposing y to vary while x and z are constant we have

$$dy = Q_1dq + R_1dr,$$
$$0 = Q_2dq + R_2dr;$$

and eliminating dr between these we have $dy = Ndq$, N being a function of p, q, r. It follows that when $dy = 0$, $dq = 0$, and therefore if we suppose z to vary while x and y are constant, we find $dz = R_2dr$, so that finally

$$dx\,dy\,dz = MNR_2dp\,dq\,dr.$$

The general expression for M is complicated, and it is of little use to give it here, as the consideration of the particular conditions of any given transformation will usually give us its value more readily than a substitution in the general formula.*

(1) Transform $x\dfrac{dR}{dy} - y\dfrac{dR}{dx}$,

having given $x = r\cos\theta$, $y = r\sin\theta$,

and therefore $x^2 + y^2 = r^2$, $\tan\theta = \dfrac{y}{x}$.

$$\frac{dR}{dx} = \frac{dR}{dr}\cos\theta - \frac{dR}{d\theta}\frac{\sin\theta}{r},$$

$$\frac{dR}{dy} = \frac{dR}{dr}\sin\theta + \frac{dR}{d\theta}\frac{\cos\theta}{r},$$

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[*] Lagrange, *Mémoires de Berlin*, 1773, p. 121.
Legendre, *Mémoires de l'Académie des Sciences*, 1788, p. 454.

whence $x\frac{dR}{dy} - y\frac{dR}{dx} = \frac{dR}{d\theta}$.

This transformation occurs in the planetary theory.

(2) Transform $x\frac{dR}{dx} + y\frac{dR}{dy}$,

the variables being the same as in the last example. The result is

$$r\frac{dR}{dr}.$$

(3) Transform $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$,

having given $x^2 + y^2 = r^2$.

$$\frac{d\phi}{dx} = \frac{d\phi}{dr}\cdot\frac{dr}{dx} = \frac{d\phi}{dr}\frac{x}{r},$$

$$\frac{d^2\phi}{dx^2} = \frac{d^2\phi}{dr^2}\frac{dr}{dx}\frac{x}{r} + \frac{d\phi}{dr}\frac{1}{r} - \frac{d\phi}{dr}\frac{dr}{dx}\frac{x}{r^2}$$

$$= \frac{d^2\phi}{dr^2}\frac{x^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right).$$

Similarly $\frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\frac{y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right)$.

Whence

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = \frac{d^2\phi}{dr^2}\frac{x^2+y^2}{r^2} + \frac{d\phi}{dr}\left(\frac{2}{r} - \frac{x^2+y^2}{r^3}\right),$$

and therefore $\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} = 0$.

This equation occurs in researches on the motion of fluids.

(4) If $\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} + \frac{d^2\phi}{dz^2} = 0$,

when $x^2 + y^2 + z^2 = r^2$,

we find
$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 0.$$

(5) Transform $\dfrac{d^2V}{dx^2} + \dfrac{d^2V}{dy^2} = 0$ into a function of r and θ, having given $x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{dV}{dy} = \sin\theta\frac{dV}{dr} + \frac{\cos\theta}{r}\frac{dV}{d\theta},$$

$$\frac{d^2V}{dy^2} = \sin^2\theta\frac{d^2V}{dr^2} + \frac{\cos^2\theta}{r^2}\frac{d^2V}{d\theta^2} + \frac{\cos^2\theta}{r}\frac{dV}{dr}$$
$$+ \frac{2\sin\theta\cos\theta}{r^2}\left(r\frac{d^2V}{dr\,d\theta} - \frac{dV}{d\theta}\right).$$

The expression for $\dfrac{d^2V}{dy^2}$ may be deduced from that of $\dfrac{d^2V}{dx^2}$ by putting $\dfrac{\pi}{2} - \theta$ for θ. We then get

$$\frac{d^2V}{dx^2} = \cos^2\theta\frac{d^2V}{dr^2} + \frac{\sin^2\theta}{r^2}\frac{d^2V}{d\theta^2} + \frac{\sin^2\theta}{r}\frac{dV}{dr}$$
$$- \frac{2\sin\theta\cos\theta}{r^2}\left(r\frac{d^2V}{dr\,d\theta} - \frac{dV}{d\theta}\right).$$

Adding these together,

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = \frac{d^2V}{dr^2} + \frac{1}{r^2}\frac{d^2V}{d\theta^2} + \frac{1}{r}\frac{dV}{dr} = 0.$$

(6) Transform $\dfrac{d^2V}{dx^2} + \dfrac{d^2V}{dy^2} + \dfrac{d^2V}{dz^2} = 0$

into a function of r, θ, and ϕ, having given

$$x = r\cos\theta,\ y = r\sin\theta\sin\phi,\ z = r\sin\theta\cos\phi.$$

A slight artifice will enable us to do this with considerable facility. Assume $\rho = r\sin\theta$, so that

$$y = \rho\sin\phi,\ z = \rho\cos\phi,$$
$$\rho = r\sin\theta,\ x = r\cos\theta.$$

Taking first the two variables y and z, we find as in the preceding example

$$\frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = \frac{d^2V}{d\rho^2} + \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} + \frac{1}{\rho}\frac{dV}{d\rho}.$$

In exactly the same way, the equations of condition being similar, we find

$$\frac{d^2V}{d\rho^2} + \frac{d^2V}{dx^2} = \frac{d^2V}{dr^2} + \frac{1}{r^2}\frac{d^2V}{d\theta^2} + \frac{1}{r}\frac{dV}{dr}.$$

Also, as in the first part of the last example,

$$\frac{1}{\rho}\frac{dV}{d\rho} = \frac{1}{r}\frac{dV}{dr} + \frac{\cot\theta}{r^2}\frac{dV}{d\theta}.$$

Adding these three expressions,

$$\frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + \frac{d^2V}{dx^2}$$

$$= \frac{d^2V}{dr^2} + \frac{1}{r^2}\frac{d^2V}{d\theta^2} + \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} + \frac{2}{r}\frac{dV}{dr} + \frac{\cot\theta}{r^2}\frac{dV}{d\theta} = 0.$$

By substituting for ρ its value, and making some obvious reductions, this becomes

$$r\frac{d^2(rV)}{dr^2} + \frac{1}{\sin^2\theta}\frac{d^2V}{d\phi^2} + \frac{d}{d.\cos\theta}\left(\sin^2\theta\frac{dV}{d.\cos\theta}\right) = 0.$$

This important equation is the basis of the Mathematical Theories of Attraction and Electricity. The artifice here used is given by Mr A. Smith in the *Cambridge Mathematical Journal*, Vol. I. p. 122.

(7) Transform the double integral

$$\iint x^{m-1}y^{n-1}\,dy\,dx$$

into one where u and v are the independent variables, x, y, u, v being connected by the equations

$$x + y = u, \quad y = uv.$$

$$\text{Here } dy\,dx = \left(\frac{dx}{du}\,\frac{dy}{dv} - \frac{dx}{dv}\,\frac{dy}{du}\right) du\,dv.$$

$$\text{Now } \frac{dx}{du} = 1, \quad \frac{dy}{dv} = u, \quad \frac{dx}{dv} = 0.$$

$$\text{Therefore } dy\,dx = u\,du\,dv,$$

$$\text{and } \iint x^{m-1}y^{n-1}dy\,dx = \iint u^{m+n-1}(1-v)^{m-1}v^{n-1}du\,dv.$$

This transformation is given by Jacobi in Crelle's *Journal*, Vol. XI. p. 307: it is of great use in the investigation of the values of definite integrals.

(8) Transform the double integral

$$\iint \epsilon^{x^2+y^2}dx\,dy$$

into one where r and θ are the independent variables, having given

$$x = r\cos\theta, \quad y = r\sin\theta,$$

$$\iint \epsilon^{x^2+y^2}dx\,dy = -\iint \epsilon^{r^2}r\,dr\,d\theta.$$

(9) Having given

$$x = r\cos\theta, \quad y = r\sin\theta\sin\phi, \quad z = r\sin\theta\cos\phi,$$

transform the triple integral

$$\iiint V\,dx\,dy\,dz$$

into a function of r, θ, and ϕ.

Using the same artifice as in Ex. 6 we find

$$\iiint V\,dx\,dy\,dz = \iiint V r^2\,dr\sin\theta\,d\theta\,d\phi.$$

This is a very important transformation, being that from rectangular to polar co-ordinates in space. If we suppose $V = 1$, $\iiint dx\,dy\,dz$ is the expression for the volume of any solid referred to rectangular co-ordinates: and it becomes $\iiint r^2dr\sin\theta\,d\theta\,d\phi$ when referred to polar co-ordinates.

(10) Having given z a function of x and y determined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

it is required to transform

$$\iint dx\, dy \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}$$

into a function of θ and ϕ when

$$x = a \sin\theta \cos\phi, \quad y = b \sin\theta \sin\phi,$$

$$\text{and consequently} \quad z = c \cos\theta.$$

In this case

$$\frac{dx}{d\theta} = a \cos\theta \cos\phi, \quad \frac{dx}{d\phi} = - a \sin\theta \sin\phi,$$

$$\frac{dy}{d\theta} = b \cos\theta \sin\phi, \quad \frac{dy}{d\phi} = b \sin\theta \cos\phi,$$

$$\frac{dz}{d\theta} = - c \sin\theta, \quad \frac{dz}{d\phi} = 0.$$

Hence

$$\frac{dx}{d\theta}\cdot\frac{dy}{d\phi} - \frac{dx}{d\phi}\,\frac{dy}{d\theta} = ab \sin\theta \cos\theta,$$

$$\frac{dz}{d\theta}\cdot\frac{dy}{d\phi} - \frac{dz}{d\phi}\,\frac{dy}{d\theta} = - bc\,(\sin\theta)^2 \cos\phi,$$

$$\frac{dz}{d\theta}\cdot\frac{dx}{d\phi} - \frac{dz}{d\phi}\,\frac{dx}{d\theta} = ac\,(\sin\theta)^2 \sin\phi.$$

Substituting these values in the general expressions for $\frac{dz}{dx}$, $\frac{dz}{dy}$, and $dx\,dy$, we find

$$\iint dx\, dy \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}$$

$$= \iint d\theta\, d\phi \sin\theta \left\{a^2 b^2 (\cos\theta)^2 + (c \sin\theta)^2 (a^2 \sin^2\phi + b^2 \cos^2\phi)\right\}^{\frac{1}{2}}.$$

Ivory, *Phil. Trans.* 1809.

CHAPTER IV.

ELIMINATION OF CONSTANTS AND FUNCTIONS BY MEANS OF DIFFERENTIATION.

Ex. (1) $y^2 = ax + b \ldots\ldots\ldots (1).$

To eliminate b, differentiate, when we have

$$2y \frac{dy}{dx} = a \ldots\ldots\ldots (2).$$

To eliminate a, substitute its value given by (2) in (1);

$$\text{then } y^2 = 2xy \frac{dy}{dx} + b.$$

To eliminate both a and b, differentiate (2) again; then

$$y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.$$

(2) Eliminate a from the equation

$$y = x^n + ae^{mx};$$

$$\frac{dy}{dx} - my = (n - mx)\, x^{n-1}.$$

(3) Eliminate a from the equation

$$y = ax + \frac{m}{a};$$

the result is $$x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} + m = 0.$$

(4) Eliminate a and b from the equation

$$y - ax^2 - bx = 0;$$

the result is $$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2y}{x^2} = 0.$$

(5) Eliminate the constants m and a from

$$y = m \cos (rx + a).$$

Differentiating twice, $\dfrac{d^2 y}{dx^2} = - r^2 m \cos (rx + a)$.

Multiplying the former by r^2 and adding,

$$\frac{d^2 y}{dx^2} + r^2 y = 0.$$

(6) Eliminate m and a from the equation

$$y^2 = m (a^2 - x^2);$$

the result is $$xy \frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx}\right)^2 - y \frac{dy}{dx} = 0.$$

(7) Eliminate c from the equation $x - y = ce^{-\frac{x}{x-y}}$.

Taking the logarithmic differential and eliminating,

$$x - 2y + y \frac{dy}{dx} = 0.$$

(8) Eliminate a and β from the equation

$$(x - a)^2 + (y - \beta)^2 = r^2.$$

Differentiating, $$(x - a) + (y - \beta) \frac{dy}{dx} = 0.$$

Differentiating again, $1 + \left(\dfrac{dy}{dx}\right)^2 + (y - \beta) \dfrac{d^2 y}{dx^2} = 0$,

whence $$y - \beta = - \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2 y}{dx^2}}, \quad x - a = \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2 y}{dx^2}} \frac{dy}{dx}.$$

Substituting these values of $y - \beta$ and $x - a$, we have

$$\frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^3}{\left(\dfrac{d^2 y}{dx^2}\right)^2} = r^2,$$

in which a and β no longer appear.

This is the expression for the square of the radius of curvature of any curve.

(9) Eliminate m from the equation

$$(\alpha + m\beta)(x^2 - my^2) = m\gamma^2;$$

the result is

$$\alpha xy \left(\frac{dy}{dx}\right)^2 + (\beta x^2 - \alpha y^2 - \gamma^2)\frac{dy}{dx} - \beta xy = 0.$$

(10) Eliminate a, b, c from the equation

$$z = ax + by + c,$$

y being a function of x.

Differentiating two and three times with respect to x,

$$\frac{d^2 z}{dx^2} = b\frac{d^2 y}{dx^2}, \quad \text{and} \quad \frac{d^3 z}{dx^3} = b\frac{d^3 y}{dx^3}.$$

Eliminating b, we have

$$\frac{d^3 z}{dx^3}\frac{d^2 y}{dx^2} - \frac{d^2 z}{dx^2}\frac{d^3 y}{dx^3} = 0.$$

This is the condition that a curve in three dimensions should be a plane curve.

(11) Eliminate the exponentials from

$$y = \frac{\epsilon^x + \epsilon^{-x}}{\epsilon^x - \epsilon^{-x}}.$$

Multiply numerator and denominator by ϵ^x, then

$$y = \frac{\epsilon^{2x} + 1}{\epsilon^{2x} - 1},$$

whence $\epsilon^{2x} = \dfrac{y+1}{y-1}$, and $2x = \log\dfrac{y+1}{y-1}$,

and differentiating, $\dfrac{dy}{dx} = 1 - y^2$.

(12) Eliminate the power from the equation

$$y = (a^2 + x^2)^{\frac{m}{n}}.$$

Taking the logarithmic differential we have

$$\frac{dy}{dx} = 2\frac{m}{n}\frac{xy}{a^2 + x^2}.$$

(13) Eliminate the functions from

$$y = \sin(\log x);$$

the result is
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0.$$

(14) Eliminate the exponential and circular functions from

$$y = a\epsilon^{mx}\sin nx.$$

Taking the logarithmic differential

$$\frac{1}{y}\frac{dy}{dx} = m + n\cot nx.$$

Differentiating again and eliminating $\cot nx$ by the last equation, we have

$$\frac{d^2y}{dx^2} - 2m\frac{dy}{dx} + (n^2 + m^2)y = 0.$$

(15) Eliminate the arbitrary function from the equation

$$z = xy\phi(y).$$

Differentiating with respect to x only,

$$\frac{dz}{dx} = y\phi(y); \qquad \text{and therefore } x\frac{dz}{dx} - z = 0.$$

(16) Eliminate the function ϕ from the equation

$$y - nz = \phi(x - mz).$$

Differentiating with respect to x only,

$$-n\frac{dz}{dx} = \phi'(x - mz)\left(1 - m\frac{dz}{dx}\right).$$

Differentiating with respect to y only,

$$1 - n\frac{dz}{dy} = -m\phi'(x - mz)\frac{dz}{dy},$$

whence
$$m\frac{dz}{dx} + n\frac{dz}{dy} = 1.$$

This is the differential equation to cylindrical surfaces.

(17) If
$$\frac{y-b}{z-c} = \phi\left(\frac{x-a}{z-c}\right),$$

by the elimination of the function we find

$$(x-a)\frac{dz}{dx} + (y-b)\frac{dz}{dy} = z - c.$$

This is the differential equation to conical surfaces.

(18) Eliminate ϕ and ψ from the equation

$$z = x^n\phi\left(\frac{y}{x}\right) + y^n\psi\left(\frac{y}{x}\right).$$

Differentiating with respect to x,

$$(1)\quad \frac{dz}{dx} = nx^{n-1}\phi\left(\frac{y}{x}\right) - yx^{n-2}\phi'\left(\frac{y}{x}\right) - \frac{y^{n+1}}{x^2}\psi'\left(\frac{y}{x}\right).$$

Differentiating with respect to y,

$$(2)\quad \frac{dz}{dy} = x^{n-1}\phi'\left(\frac{y}{x}\right) + ny^{n-1}\psi\left(\frac{y}{x}\right) + \frac{y^n}{x}\psi'\left(\frac{y}{x}\right).$$

Multiply (1) by x, (2) by y and add,

then
$$x\frac{dz}{dx} + y\frac{dz}{dy} = nz.$$

This is the differential equation to all homogeneous functions of n dimensions. It is to be observed that the two arbitrary functions are really equivalent to one only, for the original equation may be put under the form

$$z = x^n\left\{\phi\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^n\psi\left(\frac{y}{x}\right)\right\} = x^n f\left(\frac{y}{x}\right).$$

This is the reason why both functions disappear after one differentiation. If we proceeded to a second differentiation we should find

$$x^2 \frac{d^2 z}{dx^2} + 2xy \frac{d^2 z}{dx\,dy} + y^2 \frac{d^2 z}{dy^2} = n(n-1)z;$$

for the third differentiation

$$x^3 \frac{d^3 z}{dx^3} + 3x^2 y \frac{d^3 z}{dx^2 dy} + 3xy^2 \frac{d^3 z}{dx\,dy^2} + y^3 \frac{d^3 z}{dy^3} = n(n-1)(n-2)z,$$

and so on to any order. See p. 27.

(19) Eliminate the functions from the equation

$$z = \phi(x + at) + \psi(x - at),$$

x and t being variable,

$$\frac{d^2 z}{dx^2} = \phi''(x + at) + \psi''(x - at),$$

$$\frac{d^2 z}{dt^2} = a^2 \phi''(x + at) + a^2 \psi''(x - at).$$

Therefore $$\frac{d^2 z}{dt^2} - a^2 \frac{d^2 z}{dx^2} = 0.$$

This is the equation of motion for vibrating chords.

(20) Let $z = \phi\left(\frac{y^2 - x^2}{x}\right)$;

the result is $2xy \frac{dz}{dx} + (x^2 + y^2) \frac{dz}{dy} = 0.$

(21) Eliminate ϕ and ψ from the equation

$$z = x\phi(z) + y\psi(z),$$

$$\frac{dz}{dx} = \phi(z) + x\phi'(z) \frac{dz}{dx} + y\psi'(z) \frac{dz}{dx},$$

or $\frac{dz}{dx}\{1 - x\phi'(z) - y\psi'(z)\} = \phi(z).$

Similarly,

$$\frac{dz}{dy}\{1 - x\phi'(z) - y\psi'(z)\} = \psi(z).$$

Dividing the one by the other,

$$\frac{\;\frac{dz}{dx}\;}{\;\frac{dz}{dy}\;} = \frac{\phi(z)}{\psi(z)} = f(z) \text{ suppose.}$$

Differentiating with respect to x,

$$\frac{d^2z}{dx^2}\frac{dz}{dy} - \frac{dz}{dx}\frac{d^2z}{dx\,dy} = f'(z)\frac{dz}{dx}\left(\frac{dz}{dy}\right)^2.$$

Differentiating with respect to y,

$$\frac{d^2z}{dx\,dy}\frac{dz}{dy} - \frac{dz}{dx}\frac{d^2z}{dy^2} = f'(z)\left(\frac{dz}{dy}\right)^3.$$

Multiplying by $\frac{dz}{dy}$, $\frac{dz}{dx}$, and subtracting,

$$\left(\frac{dz}{dy}\right)^2\frac{d^2z}{dx^2} - 2\frac{dz}{dx}\frac{dz}{dy}\frac{d^2z}{dx\,dy} + \left(\frac{dz}{dx}\right)^2\frac{d^2z}{dy^2} = 0.$$

This is the general equation to surfaces generated by the motion of a line which constantly rests on two given lines while it remains parallel to a fixed plane.

(22) Eliminate the arbitrary functions from

$$z = \phi(ay + bx)\,.\,\psi(ay - bx).$$

Taking the logarithm we have

$$\log z = \log\phi(ay + bx) + \log\psi(ay - bx),$$

and as the functions are arbitrary their logarithms are also arbitrary functions, and we may replace them by the general characteristics F and f. Therefore, differentiating with respect to x and y successively,

$$\frac{1}{z}\frac{dz}{dx} = bF'(ay + bx) - bf'(ay - bx),$$

$$\frac{1}{z}\frac{dz}{dy} = a\,F'(ay + bx) + a\,f'(ay - bx).$$

Differentiating again,

$$\frac{1}{z}\frac{d^2z}{dx^2} - \frac{1}{z^2}\left(\frac{dz}{dx}\right)^2 = b^2 F''(ay + bx) + b^2 f''(ay - bx),$$

$$\frac{1}{z}\frac{d^2z}{dy^2} - \frac{1}{z^2}\left(\frac{dz}{dy}\right)^2 = a^2 F''(ay + bx) + a^2 f''(ay - bx).$$

Multiplying by a^2, b^2 and subtracting, we obtain as the result of the elimination of the functions

$$a^2\left\{\frac{d^2z}{dx^2} - \frac{1}{z}\left(\frac{dz}{dx}\right)^2\right\} - b^2\left\{\frac{d^2z}{dy^2} - \frac{1}{z}\left(\frac{dz}{dy}\right)^2\right\} = 0.$$

(23) Eliminate the arbitrary functions from

$$(1) \qquad x f(a) + y\,\phi(a) + z\,\psi(a) = 1,$$

where a is a function of x, y, and z given by the equation

$$(2) \qquad x f'(a) + y\,\phi'(a) + z\,\psi'(a) = 0;$$

f', ϕ', ψ' being the differential coefficients of f, ϕ, ψ.

Differentiating (1) with respect to x,

$$\{x f'(a) + y\,\phi'(a) + z\,\psi'(a)\}\,\frac{da}{dx} + f(a) + \psi(a)\,\frac{dz}{dx} = 0;$$

which by the condition (2) is reduced to

$$f(a) + \psi(a)\,\frac{dz}{dx} = 0.$$

In the same way, differentiating with respect to y, we have

$$\phi(a) + \psi(a)\,\frac{dz}{dy} = 0.$$

Since from these two equations it appears that $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are both functions of a, the one may be supposed to be a function of the other, and we may write

$$\frac{dz}{dx} = F\left(\frac{dz}{dy}\right).$$

Eliminating the function F from this equation there results

$$\left(\frac{d^2z}{dx^2}\right)\left(\frac{d^2z}{dy^2}\right) - \left(\frac{d^2z}{dx\,dy}\right)^2 = 0.$$

This is the differential equation to developable surfaces.

(24) If $u = f(x,y) = F(r,z)$, and

$$r = \phi(ax + cz) = \psi(ax - by),$$

then $$\frac{1}{a}\frac{du}{dx} + \frac{1}{b}\frac{du}{dy} + \frac{1}{c}\frac{du}{dz} = 0.$$

$$\frac{du}{dx} = \frac{du}{dr}\cdot\frac{dr}{dx} + \frac{du}{dz}\cdot\frac{dz}{dx},$$

$$\frac{du}{dy} = \frac{du}{dr}\cdot\frac{dr}{dy} + \frac{du}{dz}\cdot\frac{dz}{dy},$$

$$\frac{dr}{dx} = \left(a + c\frac{dz}{dx}\right)\phi'(ax + cz) = a\psi'(ax - by),$$

$$\frac{dr}{dz} = c\phi'(ax + cz), \qquad \frac{dr}{dy} = -b\psi'(ax - by);$$

therefore, $$\frac{1}{a}\frac{dr}{dx} + \frac{1}{b}\frac{dr}{dy} = 0,$$

and $$\frac{1}{a}\frac{du}{dx} + \frac{1}{b}\frac{du}{dy} = \frac{du}{dz}\left(\frac{1}{a}\frac{dz}{dx} + \frac{1}{b}\frac{dz}{dy}\right).$$

Also $$\left(a + c\frac{dz}{dx}\right)\phi'(ax + cz) = a\psi'(ax - by),$$

$$c\frac{dz}{dy}\phi'(ax + cz) = -b\psi'(ax - by),$$

whence $$\frac{1}{a}\frac{dz}{dx} + \frac{1}{b}\frac{dz}{dy} = -\frac{1}{c};$$

and therefore $$\frac{1}{a}\frac{du}{dx} + \frac{1}{b}\frac{du}{dy} + \frac{1}{c}\frac{du}{dz} = 0.$$

CHAPTER V.

APPLICATION OF THE DIFFERENTIAL CALCULUS TO THE DEVELOPMENT OF FUNCTIONS.

SEC. 1. *Taylor's Theorem.*

THIS theorem, the most important in the Differential Calculus, and the foundation of the other theorems for the development of Functions, was first given by Brook Taylor in his *Methodus Incrementorum*, p. 23. He introduces it merely as a corollary to the corresponding theorem in Finite Differences, and makes no application of it, or remark on its importance. The following is the statement of the theorem:

If $u = f(x)$ and x receive an increment h, then

$$f(x+h) = u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1.2} + \frac{d^3u}{dx^3}\frac{h^3}{1.2.3} + \&c.$$

If we avail ourselves of the method of the separation of the symbols of operation from those of quantity, this theorem may be expressed in a very convenient form, which is useful in various parts of the Integral Calculus: viz.

$$f(x+h) = \left\{1 + h\frac{d}{dx} + \frac{h^2}{1.2}\frac{d^2}{dx^2} + \frac{h^3}{1.2.3}\frac{d^3}{dx^3} + \&c.\right\} f(x)$$

$$= \epsilon^{h\frac{d}{dx}} f(x).$$

It is frequently convenient to use Lagrange's notation, and to represent the successive differential coefficients of $f(x)$ by accents affixed to the characteristic of the function. In this way Taylor's Theorem is written

$$f(x+h) = f(x) + f'(x)\,h + f''(x)\frac{h^2}{1.2} + f'''(x)\frac{h^3}{1.2.3} + \&c.$$

If we stop at any term, as the nth, which is $f^{(n-1}(x)\dfrac{h^{n-1}}{1\,.\,2\,\ldots\,(n-1)}$, the error committed by neglecting the remaining terms lies between the greatest and least values which $f^{(n)}(x+\theta h)\dfrac{h^n}{1\,.\,2\,\ldots\,n}$ can receive; where θ is less than 1. This is Lagrange's Theorem of the limits of Taylor's Theorem. See Lagrange, *Calcul des Fonctions*, p. 88. Also De Morgan's *Differential Calculus*, p. 70.

Ex. (1) Let $f(x) = (a+x)^n$. Then

$$(a+x+h)^n = (a+x)^n + n(a+x)^{n-1}h + \frac{n(n-1)}{1\,.\,2}(a+x)^{n-2}h^2 + \&c.$$

(2) Let $f(x) = a^x$. Then as $\dfrac{d^n}{dx^n}a^x = (\log a)^n a^x$,

$$a^{x+h} = a^x\left\{1 + (\log a)\,h + (\log a)^2\frac{h^2}{1\,.\,2} + (\log a)^3\frac{h^3}{1\,.\,2\,.\,3} + \ldots\right\}.$$

If we stop at the nth term the error lies between the greatest and least values of $a^{(x+\theta h)}(\log a)^n\dfrac{h^n}{1\,.\,2\,\ldots\,n}$. The least value is found by making $\theta = 0$, and the greatest by making $\theta = 1$, and therefore the error lies between

$$a^{x+h}(\log a)^n\frac{h^n}{1\,.\,2\ldots n}, \quad \text{and} \quad a^x(\log a)^n\frac{h^n}{1\,.\,2\ldots n}.$$

(3) Let $f(x) = \log x$. Then since by Chap. II. Sec. 1. Ex. 11,

$$\frac{d^r}{dx^r}(\log x) = (-)^{r-1}\frac{(r-1)(r-2)\ldots2\,.\,1}{x^{r-1}},$$

$$\log(x+h) = \log x + \frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3} - \&c.,$$

and the error of stopping at the nth term lies between

$$\pm\frac{h^n}{nx^n} \quad \text{and} \quad \pm\frac{h^n}{n(x+h)^n}.$$

(4) Let $f(x) = \dfrac{1+x}{1-x}$. Then since by Chap. II. Sec. 1. Ex. 12,

$$\frac{d^r}{dx^r}\,\frac{1+x}{1-x} = 2\,\frac{r(r-1)\ldots 3\,.\,2}{(1-x)^{r+1}}$$

$$\frac{1+x+h}{1-x-h} = \frac{1+x}{1-x} + 2\left\{\frac{h}{(1-x)^2} + \frac{h^2}{(1-x)^3} + \frac{h^3}{(1-x)^4} + \&c.\right\}.$$

(5) Let $f(x) = \epsilon^{ax}\cos nx$. Then as by Chap. II. Sec. 1. Ex. 10,

$$\frac{d^r}{dx^r}\,(\epsilon^{ax}\cos nx) = (a^2+n^2)^{\frac{r}{2}}\,\epsilon^{ax}\cos(nx + r\phi),$$

$\left(\text{where } \phi = \tan^{-1}\dfrac{n}{a}\right)$,

$$\epsilon^{a(x+h)}\cos n(x+h) = \epsilon^{ax}\left\{\cos nx + (a^2+n^2)^{\frac{1}{2}}\cos(nx+\phi)\,.\,h\right.$$
$$\left. + (a^2+n^2)\cos(nx+2\phi)\,\frac{h^2}{1.2} + (a^2+n^2)^{\frac{3}{2}}\cos(nx+3\phi)\,\frac{h^3}{1.2.3} + \&c.\right\}$$

If $a = \cos\theta$, $n = \sin\theta$,

$$\epsilon^{(x+h)\cos\theta}\cos\{(x+h)\sin\theta\} = \epsilon^{x\cos\theta}\left\{\cos(x\sin\theta) + h\cos(x\sin\theta+\theta)\right.$$
$$\left. + \frac{h^2}{1\,.\,2}\cos(x\sin\theta + 2\theta) + \frac{h^3}{1\,.\,2\,.\,3}\cos(x\sin\theta + 3\theta) + \&c.\right\}.$$

(6) If $f(x) = \tan^{-1}x$, and we put

$$\frac{1}{1+x^2} = \sin y, \text{ or } \tan^{-1}x = \frac{\pi}{2} - y,$$

we have, by Chap. II. Sec. 1. Ex. 24,

$$\left(\frac{d}{dx}\right)^r \tan^{-1}x = (-)^{r-1}\,.\,(r-1)\,(r-2)\ldots 2\,.\,1\sin ry\,.\,(\sin y)^r,$$

therefore

$$\tan^{-1}(x+h) = \tan^{-1}x + \sin y\sin y\,\frac{h}{1} - \sin 2y\,(\sin y)^2\,\frac{h^2}{2}$$
$$+ \sin 3y\,(\sin y)^3\,\frac{h^3}{3} - \sin 4y\,(\sin y)^4\,\frac{h^4}{4} + \&c.$$

From this development Euler* has deduced many remarkable theorems, some of which are subjoined.

In the preceding example let $h = -x$, then

$$\tan^{-1}(x+h) = \tan^{-1} 0 = 0;$$

therefore $\tan^{-1} x = \sin y \,.\, \sin y \,.\, x + (\sin y)^2 \sin 2y \frac{x^2}{2}$

$$+ (\sin y)^3 \sin 3y \frac{x^3}{3} + (\sin y)^4 \sin 4y \frac{x^4}{4} + \&c.$$

Now $\tan^{-1} x = \frac{\pi}{2} - y$, and $x = \cot y = \frac{\cos y}{\sin y}$;

therefore

$$\frac{\pi}{2} = y + \sin y \cos y + \tfrac{1}{2} \sin 2y (\cos y)^2 + \tfrac{1}{3} \sin 3y (\cos y)^3 + \&c.$$

Again, let $h = -\left(x + \frac{1}{x}\right) = -\frac{1}{\sin y \cos y}$; then

$$\tan^{-1}(x+h) = \tan^{-1}\left(-\frac{1}{x}\right) = -\tan^{-1}\frac{1}{x} = -\frac{\pi}{2} + \tan^{-1} x;$$

therefore

$$\frac{\pi}{2} = \frac{\sin y}{\cos y} + \tfrac{1}{2}\frac{\sin 2y}{(\cos y)^2} + \tfrac{1}{3}\frac{\sin 3y}{(\cos y)^3} + \tfrac{1}{4}\frac{\sin 4y}{(\cos y)^4} + \&c.$$

Again, let $h = -(1+x^2)^{\frac{1}{2}} = -\frac{1}{\sin^2 y}$; then

$$\tan^{-1}\{x - (1+x^2)^{\frac{1}{2}}\} = \tan^{-1}\left(\frac{\cos y - 1}{\sin y}\right) = -\tan^{-1}\left(\tan\frac{y}{2}\right) = -\frac{y}{2};$$

therefore, as $\tan^{-1} x = \frac{\pi}{2} - y$,

$$\frac{\pi}{2} = \frac{y}{2} + \sin y + \tfrac{1}{2} \sin 2y + \tfrac{1}{3} \sin 3y + \&c.$$

If we differentiate this series we find

$$0 = \tfrac{1}{2} + \cos y + \cos 2y + \cos 3y + \&c.$$

In these formulæ y lies between 0 and $\frac{1}{2}\pi$.

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* *Calc. Diff.* p. 380.

(7) Let $u = \cot^{-1} x$, then $\cot^{-1}(x+h)$ is easily found from the expression for $\tan^{-1}(x+h)$. For since

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x, \quad \frac{du}{dx} = -\frac{1}{1+x^2},$$

and we have merely to substitute $\cot^{-1} x$ for $\tan^{-1} x$ and to change the signs of the terms beginning with the second: and as in this case $y = u$, we find

$$\cot^{-1}(x+h) = u - \sin u \sin u \frac{h}{1} + (\sin u)^2 \sin 2u \frac{h^2}{2} - \&c.$$

SEC. 2. *Maclaurin's or Stirling's Theorem.*

This Theorem, which is usually called Maclaurin's, but which ought to bear the name of Stirling, was first given by James Stirling in his *Lineæ Tertii Ordinis Newtonianæ*, p. 32. Maclaurin introduced it into his *Treatise of Fluxions*, p. 610, and his name has generally been given to the theorem from an erroneous idea that his work was the first in which it appeared.

The following is the enunciation of the Theorem:

If $f(x)$ be a function of x, and if we represent the values which it and its successive differential coefficients acquire when $x = 0$, by $f(0)$, $f'(0)$, $f''(0)$, $f'''(0)$, &c; then

$$f(x) = f(0) + f'(0)\frac{x}{1} + f''(0)\frac{x^2}{1.2} + f'''(0)\frac{x^3}{1.2.3} + \&c.$$

This Theorem is evidently a particular case of that of Taylor.

Ex. (1) Let $u = f(x) = (1+x)^{\frac{1}{2}}$; $f(0) = 1$,

$$\frac{du}{dx} = \frac{1}{2}\frac{1}{(1+x)^{\frac{1}{2}}}, \quad f'(0) = \frac{1}{2},$$

$$\frac{d^2u}{dx^2} = -\frac{1}{2^2}\frac{1}{(1+x)^{\frac{3}{2}}}, \quad f''(0) = -\frac{1}{2^2},$$

$$\frac{d^3u}{dx^3} = +\frac{3}{2^3}\,\frac{1}{(1+x)^{\frac{5}{2}}}, \quad f'''(0) = \frac{3}{2^3},$$

$$\&c. \qquad\qquad \&c.$$

Therefore,

$$(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}x - \frac{1}{2^2}\,\frac{x^2}{1.2} + \frac{3}{2^3}\,\frac{x^3}{1.2.3} - \frac{3.5}{2^4}\,\frac{x^4}{1.2.3.4} + \&c.$$

(2) Let $u = (1 + 2x + 3x^2)^{-\frac{1}{2}}$.

Then by the formula (B), Chap. II. Sec. 1,

$$\frac{d^ru}{dx^r} = (-)^r\, 1.2 \ldots r\, (1+3x)^r\, (1+2x+3x^2)^{-r+\frac{1}{2}} \times$$

$$\left\{1 - \frac{1}{2}\,\frac{r(r-1)}{1.2}\,\frac{2}{(1+3x)^2} + \frac{1.3}{2^3}\,\frac{r(r-1)\ldots(r-3)}{1.2.3.4}\cdot\frac{4}{(1+3x)^4} - \&c.\right\}$$

Whence we find

$$f(0) = 1, \quad f'(0) = -1, \quad f''(0) = 1.2\,(1-1) = 0,$$

$$f'''(0) = -1.2.3\,(1-3) = 1.2.3.2,$$

$$f^{iv}(0) = -1.2.3.4.\frac{7}{2}, \quad f^{v}(0) = 1.2.3.4.5.\frac{3}{2}.$$

Therefore,

$$(1+2x+3x^2)^{-\frac{1}{2}} = 1 - x + 2x^3 - \frac{7}{2}x^4 + \frac{3}{2}x^5 - \&c.$$

(3) Let $u = \cos x$,

$$\text{then as } \frac{d^ru}{dx^r} = \cos\left(x + r\,\frac{\pi}{2}\right),$$

$$u = \cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \&c.$$

(4) Let $u = \sin^{-1} x$.

Then by Chap. II. Sec. 1. Ex. 23,

$$\frac{d^r u}{dx^r} = \frac{1.2\ldots(r-1)\,x^{r-1}}{(1-x^2)^{r-\frac{1}{2}}}\left\{1 + \frac{1}{2}\,\frac{(r-1)(r-2)}{1.2}\,\frac{1}{x^2}\right.$$

$$\left. + \frac{1.3}{2.4}\,\frac{(r-1)(r-2)(r-3)(r-4)}{1.2.3.4}\,\frac{1}{x^4} + \&c.\right\}$$

Therefore,

$$f(0) = 0, \quad f'(0) = 1,$$

$$f''(0) = 0, \quad f'''(0) = \frac{1}{2}.1.2,$$

$$f^{iv}(0) = 0, \quad f^{v}(0) = \frac{1.3}{2.4}.1.2.3.4,$$

$$f^{vi}(0) = 0, \quad f^{vii}(0) = \frac{1.3.5}{2.4.6}.1.2.3.4.5.6.$$

Whence

$$\sin^{-1}x = x + \frac{1}{2}\,\frac{x^3}{3} + \frac{1.3}{2.4}\,\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\,\frac{x^7}{7} + \&c.$$

It was by means of this series that Newton calculated the value of π. *Commercium Epistolicum*, p. 85, 2nd Edit.

(5) Let $u = \tan^{-1}(x)$.

By means of Chap. II. Sec. 1. Ex. 24, we find

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \&c.$$

This is Gregorie's series. See *Commercium Epistolicum*, p. 98, 2nd Edit.

(6) Let $u = \sec x$;

then $f(0) = 1$, $f'(0) = 0$, $f''(0) = 1$,

$f'''(0) = 0$, $f^{iv}(0) = 5$, $f^{v}(0) = 0$, $f^{vi}(0) = 61$.

Therefore,

$$\sec x = 1 + \frac{x^2}{1.2} + \frac{5x^4}{1.2.3.4} + \frac{61x^6}{1.2.3.4.5.6} + \&c.$$

James Gregorie, *Ib.* p. 99.

(7) Let $u = \tan x$;

then $f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = 2,$

$$f^{iv}(0) = 0, \quad f^{v}(0) = 16, \quad f^{vi}(0) = 0, \quad f^{vii}(0) = \frac{816}{9},$$

Therefore, $\tan x = x + \dfrac{x^3}{1\,.\,3} + \dfrac{2x^5}{3\,.\,5} + \dfrac{17x^7}{3\,.\,5\,.\,7\,.\,9} + \text{\&c.}$

James Gregorie, *Ib.*

(8) Let $u = \epsilon^{ax} \cos nx$. Then

$$\epsilon^{ax} \cos nx = 1 + (a^2 + n^2)^{\frac{1}{2}} \cos \phi \frac{x}{1} + (a^2 + n^2) \cos 2\phi \frac{x^2}{1\,.\,2}$$

$$+ (a^2 + n^2)^{\frac{3}{2}} \cos 3\phi \frac{x^3}{1\,.\,2\,.\,3} + (a^2 + n^2)^2 \cos 4\phi \frac{x^4}{1\,.\,2\,.\,3\,.\,4} + \text{\&c.}$$

If $a = \cos \theta, \quad n = \sin \theta, \quad \phi = \theta, \quad a^2 + n^2 = 1,$

$$\epsilon^{x \cos \theta} \cos (x \sin \theta) = 1 + x \cos\theta + \frac{x^2}{1\,.\,2} \cos 2\theta + \frac{x^3}{1\,.\,2\,.\,3} \cos 3\theta + \text{\&c.}$$

If $a = n = 1, \quad a^2 + n^2 = 2, \quad \phi = \dfrac{\pi}{4},$

$$\cos \frac{\pi}{4} = \frac{1}{2^{\frac{1}{2}}}, \quad \cos 2\frac{\pi}{4} = 0, \quad \cos 3\frac{\pi}{4} = -\frac{1}{2^{\frac{1}{2}}}, \quad \cos 4\frac{\pi}{4} = -1,$$

$$\cos 5\frac{\pi}{4} = -\frac{1}{2^{\frac{1}{2}}}, \quad \cos 6\frac{\pi}{4} = 0, \quad \cos 7\frac{\pi}{4} = \frac{1}{2^{\frac{1}{2}}}, \quad \cos 8\frac{\pi}{4} = 1, \quad \text{\&c.};$$

therefore,

$$\epsilon^{x} \cos x = 1 + \frac{x}{1} - 2\,.\,\frac{x^3}{1\,.\,2\,.\,3} - \frac{2^2 x^4}{1\,.\,2\,.\,3\,.\,4} - \frac{2^2 x^5}{1\,.\,2\,.\,3\,.\,4\,.\,5}$$

$$+ \frac{2^3 x^7}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7} + \frac{2^4 x^8}{1\,.\,2\,.\,.\,.\,.\,7\,.\,8} + \text{\&c.}$$

If $a = \frac{1}{2}, \quad n = \dfrac{3^{\frac{1}{2}}}{2}, \quad a^2 + n^2 = 1, \quad \phi = \dfrac{\pi}{3}$;

$$\epsilon^{\frac{x}{2}} \cos \frac{x 3^{\frac{1}{2}}}{2} = 1 + \frac{1}{2}\frac{x}{1} - \frac{1}{2}\frac{x^2}{1\,.\,2} - \frac{x^3}{1\,.\,2\,.\,3} - \frac{1}{2}\frac{x^4}{1\,.\,2\,.\,3\,.\,4}$$

$$+ \frac{1}{2}\frac{x^5}{1\,.\,2\,.\,.\,5} + \frac{x^6}{1\,.\,2\,.\,.\,.\,6} + \frac{1}{2}\frac{x^7}{1\,.\,2\,.\,.\,.\,7} + \text{\&c.}$$

(9) Let $u = (1 + \epsilon^x)^n$,

$$f(0) = 2^n, \; f'(0) = n2^{n-1}, \; f''(0) = n2^{n-2}(n+1),$$

$$f'''(0) = n^2 2^{n-3}(n+3), \; f^{IV}(0) = n2^{n-4}(n^3+6n^2+3n-2);$$

therefore,

$$(1+\epsilon^x)^n = 2^n\left\{1 + \frac{n}{2}\frac{x}{1} + \frac{n(n+1)}{2^2}\frac{x^2}{1.2} + \frac{n^2(n+3)}{2^3}\frac{x^3}{1.2.3} + \frac{n(n^3+6n^2+3n-2)}{2^4}\frac{x^4}{1.2.3.4} + \&c.\right\}.$$

If $n = \frac{1}{2}$, then

$$(1+\epsilon^x)^{\frac{1}{2}} = 2^{\frac{1}{2}}\left\{1 + \frac{1}{2^2}\frac{x}{1} + \frac{3}{2^4}\frac{x^2}{1.2} + \frac{7}{2^6}\frac{x^3}{1.2.3} + \frac{9}{2^8}\frac{x^4}{1.2.3.4} + \&c.\right\}.$$

Maclaurin's Theorem may also be applied to the development of implicit functions, the differentiations being effected by the methods required in such cases.

(10) Let $u^2 - ux - 1 = 0$.

Expand u in terms of x.

When $x = 0$, $u^2 = 1$; therefore $f(0) = \pm 1$.

Differentiating the implicit function we have

$$2u\frac{du}{dx} - x\frac{du}{dx} - u = 0;$$

when $x = 0$, $u = \pm 1$, therefore $f'(0) = \frac{1}{2}$.

Differentiating again,

$$\left(2\frac{du}{dx} - 1\right)\frac{du}{dx} + (2u - x)\frac{d^2u}{dx^2} - \frac{du}{dx} = 0,$$

when $x = 0$, $\frac{du}{dx} = \frac{1}{2}$, $u = \pm 1$; therefore $f''(0) = \pm\frac{1}{4}$.

Differentiating again,

$$3\left(2\frac{du}{dx} - 1\right)\frac{d^2u}{dx^2} + (2u - x)\frac{d^3u}{dx^3} = 0;$$

when $x = 0$, $2\dfrac{du}{dx} - 1 = 0$, therefore $f'''(0) = 0$.

In the same way we should find,

$$f^{iv}(u) = \mp \frac{1^2 . 3}{2^4}, \quad f^{v}(u) = 0, \quad f^{vi}(u) = \pm \frac{1^2 . 3^2 . 5}{2^6}, \text{ \&c.};$$

therefore,

$$u = \pm 1 + \frac{1}{2}\frac{x}{1} \pm \frac{1}{2^2}\frac{x^2}{1 . 2} \mp \frac{1^2 . 3}{2^4}\frac{x^4}{1.2.3.4} \pm \frac{1^2 . 3^2 . 5}{2^6}\frac{x^5}{1.2.3.4.5} + \text{\&c.}$$

Since the given function is a quadratic in u it involves really two different functions of x, which in the development are given by means of the double sign.

(11) Let $\quad u^3 - 6ux - 8 = 0$,

When $x = 0$, $\quad u^3 = 8$, $\quad u = (8)^{\frac{1}{3}}$.

The possible root of this is 2, and if we take it, we find by the same method as in the last example the series

$$u = 2 + x - \frac{1}{2}\frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} + \text{\&c.}$$

The other series for u would be found by taking the impossible values of the cube root of 8.

(12) Let $u^3 - a^2u + axu - x^3 = 0$.

When $x = 0$, $u^3 - a^2u = 0$, which gives

$$u = 0, \quad u = \pm a.$$

Taking the first of these values, we find the series

$$u = -\frac{x^3}{a^2} - \frac{x^4}{a^3} - \frac{x^5}{a^4} - \text{\&c.}$$

Taking the positive value of a,

$$u = a - \frac{x}{2} - \frac{x^2}{8a} + \frac{3x^3}{16a^2}, \text{ \&c.}$$

Taking the negative value of a,

$$u = -a + \frac{x}{2} + \frac{x^2}{8a} + \frac{5x^3}{8a^2}, \text{ \&c.}$$

(13) If $\sin y = x \sin (a + y)$,
expand y in terms of x.

When $x = 0$, $\sin y = 0$; therefore $y = r\pi$, r being 0, or any positive integer.

Differentiating, $\cos y \dfrac{dy}{dx} = \sin (a + y) + x \cos (a + y) \dfrac{dy}{dx}$;

putting $x = 0$, $y = r\pi$, we have

$$f'(0) = \frac{\sin (a + r\pi)}{\cos r\pi} = \sin a.$$

Differentiating again,

$$\cos y \frac{d^2y}{dx^2} - \sin y \left(\frac{dy}{dx}\right)^2 = 2 \cos (a + y) \frac{dy}{dx}$$

$$- x \sin (a + y) \left(\frac{dy}{dx}\right)^2 + x \cos (a + y) \frac{d^2y}{dx^2}.$$

$$f''(0) = 2 \sin a \frac{\cos (a + r\pi)}{\cos r\pi} = 2 \sin a \cos a = \sin 2a.$$

In a similar manner we should find

$$f'''(0) = 2 \sin a \left\{3 - 4 (\sin a)^2\right\},$$

and so on; therefore, substituting in Maclaurin's Theorem,

$$y = r\pi + \sin a \frac{x}{1} + \sin 2a \frac{x^2}{1.2} + 2 \sin a \left\{3 - 4(\sin a)^2\right\} \frac{x^3}{1.2.3} + \&c.$$

(14) If $u^n \log u = ax$, expand u in terms of x.

When $x = 0$, one value of u is 1, as $\log 1 = 0$; therefore taking $f(0) = 1$, we find

$$f'(0) = a, \quad f''(0) = -(2n - 1) a^2, \quad f'''(0) = (3n - 1)^2 a^3,$$

$$f^{\text{iv}}(0) = -(4n - 1)^3 a^4, \ \&c.$$

Hence we have

$$u = 1 + ax - (2n - 1) \frac{a^2x^2}{1.2} + (3n - 1)^2 \frac{a^3x^3}{1.2.3} - (4n - 1)^3 \frac{a^4x^4}{1.2.3.4} + \&c.$$

(15) Let $y = 1 + x\epsilon^y$ expand y in terms of x.

Here $f(0) = 1$, $f'(0) = \epsilon$, $f''(0) = 2\epsilon^2$,

$f'''(0) = 9\epsilon^3$, $f^{\text{iv}}(0) = 64\epsilon^4$.

Therefore,

$$y = 1 + \epsilon . x + 2\epsilon^2 \frac{x^2}{1.2} + 9\epsilon^3 \frac{x^3}{1.2.3} + 64\epsilon^4 \frac{x^4}{1.2.3.4} + \&c.$$

As the calculation of the high differential coefficients of implicit functions is necessarily very tedious, this application of Maclaurin's Theorem is not of much use; and a better means of expanding implicit functions, is to be found in the Theorems of Lagrange and Laplace, to which we now proceed.

SEC. 3. *Theorems of Lagrange and Laplace.*

If y be given in an equation of the form

$$y = z + x\phi(y),$$

and if $u = f(y)$, f and ϕ being any functions whatever, then u may be expanded in ascending powers of x by the theorem

$$u = f(z) + \{\phi(z) f'(z)\} \frac{x}{1} + \frac{d}{dz}[\{\phi(z)\}^2 f'(z)] \frac{x^2}{1.2}$$

$$+ \frac{d^2}{dz^2}[\{\phi(z)\}^3 f'(z)] \frac{x^3}{1.2.3} + \frac{d^3}{dz^3}[\{\phi(z)\}^4 f'(z)] \frac{x^4}{1.2.3.4} + \&c.$$

This is Lagrange's Theorem. See *Equations Numériques, Note XI; Mémoires de Berlin*, 1768, p. 251.

The Theorem of Laplace is an extension of the preceding made by assuming the given equation in y to be

$$y = F\{z + x\phi(y)\}.$$

Then if $u = f(y)$, and if we put $fF(z) = f_1(z)$, and $\frac{d}{dz} fF(z) = f_1'(z)$, and $\phi F(z) = \phi_1(z)$,

$$u = f(y) = f_1(z) + \phi_1(z) f_1'(z) \frac{x}{1} + \frac{d}{dz}[\{\phi_1 z\}^2 f_1'(z)] \frac{x^2}{1.2} + \&c.$$

Mémoires de l'Académie des Sciences, 1777, p. 99.

In these theorems, if we make $f(y) = y$, we find

$$y = z + \phi(z)\frac{x}{1} + \frac{d}{dz}\{\phi(z)\}^2\frac{x^2}{1.2} + \frac{d^2}{dz^2}\{\phi(z)\}^3\frac{x^3}{1.2.3} + \&c.$$

and $y = F(z) + \phi_1(z)F(z)\frac{x}{1} + \frac{d}{dz}[\{\phi_1(z)\}^2F'(z)]\frac{x^2}{1.2} + \&c.$

Ex. (1) Let $y^3 - ay + b = 0$, or $y = \frac{b}{a} + \frac{1}{a}y^3$;

Expand y in ascending powers of $\frac{1}{a}$.

Here $f(y) = y$, $\phi(y) = y^3$, $z = \frac{b}{a}$.

Therefore $\frac{d}{dz}\{\phi(z)\}^2 = 6\left(\frac{b}{a}\right)^5$, $\frac{d^2}{dz^2}\{\phi(z)\}^3 = 9.8.\left(\frac{b}{a}\right)^7 + \&c$

Whence $y = \frac{b}{a}\left(1 + \frac{b^2}{a^3} + 3\frac{b^4}{a^6} + 12\frac{b^6}{a^9} + 55\frac{b^8}{a^{12}} + \&c.\right)$

(2) Let $a - y + by^n = 0$, or $y = a + by^n$.

Expand y in terms of b.

Here $f(y) = y$, $\phi(y) = y^n$, $z = a$. Then

$$y = a\left\{1 + a^{n-1}b + 2n.a^{2n-2}\frac{b^2}{1.2} + 3n(3n-1)a^{3n-3}\frac{b^3}{1.2.3} + \&c.\right\}.$$

(3) Let $b - y + ca^y = 0$, or $y = b + ca^y$.

Expand y in terms of c.

Here $f(y) = y$, $\phi(y) = a^y$, $z = b$. Then

$$y = b + a^b.\frac{c}{1} + 2\log a\, a^{2b}\frac{c^2}{1.2} + 3^2(\log a)^2a^{3b}\frac{c^3}{1.2.3} + \&c.$$

If $b = 1$, or $y = 1 + ca^y$,

$$y = 1 + a\frac{c}{1} + 2\log a\frac{a^2c^2}{1.2} + 3^2(\log a)^2\frac{a^3c^3}{1.2.3} + \&c.$$

See Ex. 15 of the preceding Section.

(4) Let $y = a + x \log y$.

Expand y in terms of x.

Here $f(z) = z$, $f'(z) = 1$, $z = a$, $\phi(z) = \log z$. Therefore

$$y = a + \log a \cdot \frac{x}{1} + \frac{2 \log a}{a} \frac{x^2}{1.2} + \frac{3 \log a}{a^2} (2 - \log a) \frac{x^3}{1.2.3}$$

$$+ \frac{4 \log a}{a^3} \{6 - 9 (\log a)^2 + 2 (\log a)^3\} \frac{x^4}{1.2.3.4} + \&c.$$

(5) Let $a - y + b(y^n + cy^r) = 0$,

$$\text{or} \quad y = a + b(y^n + cy^r):$$

expand y in terms of b.

Here $\phi(z) = z^n + cz^r$, $z = a$; therefore

$$y = a + (a^n + ca^r) \frac{b}{1} + \{2na^{2n-1} + 2c(n+r) a^{n+r-1} + c^2 2r a^{2r-1}\} \frac{b^2}{1.2}$$

$$+ \&c.$$

In the preceding examples it will be seen that the expansion of y in terms of b is the solution of an equation either algebraic or transcendental, and Lagrange has shown that the series always gives the *least* root of the equation.

(6) Let $y^3 - ay + b = 0$:

expand y^n in terms of $\frac{1}{a}$.

Here $f(z) = z^n$, $\phi(z) = z^3$, $z = \frac{b}{a}$.

Whence

$$y^n = \frac{b^n}{a^n} \left\{ 1 + n \frac{b^2}{a^2} \frac{1}{a} + \frac{n(n+5)}{1.2} \frac{b^4}{a^4} \frac{1}{a^2} + \frac{n(n+7)(n+8)}{1.2.3} \frac{b^6}{a^6} \frac{1}{a^3} \right.$$

$$\left. + \frac{n(n+9)(n+10)(n+11)}{1.2.3.4} \cdot \frac{b^8}{a^8} \frac{1}{a^4} + \&c. \right\}.$$

(7) Let $1 - y + ay^r = 0$:

expand y^n in terms of a.

$$y^n = 1 + \frac{n}{1}a + \frac{n(n+2r-1)}{1.2}a^2 + \frac{n(n+3r-1)(n+3r-2)a^3}{1.2.3}$$

$$+ \frac{n(n+4r-1)(n+4r-2)(n+4r-3)}{1.2.3.4}a^4 + \&c.$$

(8) Let $1 - y + a\epsilon^y = 0$;

expand y^n in terms of a,

$$y^n = 1 + n\epsilon . a + \frac{n(n+1)}{1.2}\epsilon^2 a^2 + \frac{n(n^2+3n+5)}{1.2.3}\epsilon^3 a^3 + \&c.$$

Lagrange has shown* that if by his theorem we develop the n^{th} negative power of the root of the equation

$$y = z + x\phi(y),$$

and if we only retain the terms involving negative powers of z, the result gives us the sum of the n^{th} negative powers of the roots; while, as has just been stated, the whole series gives the n^{th} negative power of the least root.

(9) If the equation be

$$cy^2 - by + a = 0,$$

of which the two roots are α, β, then

$$\frac{1}{\alpha^n} + \frac{1}{\beta^n} = \left(\frac{b}{a}\right)^4 \Big\{1 - \frac{nc}{b}.\frac{a}{b} + \frac{n(n-3)}{1.2}\frac{c^2}{b^2}.\left(\frac{a}{b}\right)^2$$

$$- \frac{n(n-4)(n-5)}{1.2.3}\frac{c^3}{b^3}\left(\frac{a}{b}\right)^3 + \&c.\Big\},$$

the series only continuing so long as there are positive powers of $\frac{b}{a}$, that is, negative powers of $\frac{a}{b}$ or z.

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* *Equations Numériques*, p. 225.

(10) Let $a - by + cy^r = 0$.

Then if we represent the sum of the inverse n^{th} powers of the roots by $\Sigma(a^{-n})$, we have

$$\Sigma(a^{-n}) = \left(\frac{b}{a}\right)^n \left\{1 - n\left(\frac{a}{b}\right)^{r-1} . \frac{c}{b} + \frac{n(n-2r+1)}{1.2}\left(\frac{a}{b}\right)^{2r-2}\frac{c^2}{b^2}\right.$$

$$\left. - \frac{n(n-3r+1)(n-3r+2)}{1.2.3}\left(\frac{a}{b}\right)^{3r-3}\frac{c^3}{b^3} + \&c.\right\},$$

the series being continued only so long as it involves positive powers of $\frac{b}{a}$.

If in these equations we substitute $\frac{1}{y}$ for y, and then find the sum of the inverse n^{th} powers of the roots of the transformed equation, we obtain a series for the direct n^{th} powers of the roots of the original equation.

(11) If we thus transform the equation in Ex. 10 it becomes

$$c - by + ay^2 = 0;$$

and if α, β be the same quantities as before,

$$\alpha^n + \beta^n = \left(\frac{b}{c}\right)^n \left\{1 - n\frac{c}{b}\frac{a}{b} + \frac{n(n-3)}{1.2}\frac{c^2}{b^2}.\frac{a^2}{b^2} + \&c.\right\},$$

continued so long as there are positive powers of $\frac{b}{c}$.

(12) Let $u = m + e\sin u$.

Expand u and $\sin u$ in terms of e.

The expression for u is

$$u = m + \sin m . \frac{e}{1} + \sin 2m \frac{e^2}{1.2} + \frac{3}{4}(3\sin 3m - \sin m)\frac{e^3}{1.2.3}$$

$$+ (8\sin 4m - 4\sin 2m)\frac{e^4}{1.2.3.4} + \&c.$$

The expression for sin u is

$$\sin u = \sin m + \frac{\sin 2m}{2}\frac{e}{1} + \frac{3\sin 3m - \sin m}{4}\frac{e^2}{1\,.\,2}$$

$$+ (2\sin 4m - \sin 2m)\frac{e^3}{1\,.\,2\,.\,3} + \&c.$$

(13) We might employ Lagrange's Theorem to express h in terms of u from the equation

$$u + \frac{du}{dx}h + \frac{d^2u}{dx^2}\frac{h^2}{1\,.\,2} + \frac{d^3u}{dx^3}\frac{h^3}{1\,.\,2\,.\,3} + \&c. = 0;$$

but the following method is more convenient, as it gives at once the law of the series. It is easily seen that the series is the development of some function of $x + h$, which when $h = 0$ becomes u.

Let $u = f(x)$, then $f(x + h) = 0$. But since $u = f(x)$, $x = f^{-1}(u)$, and if we call k the increment of u due to the increment h of x,

$$x + h = f^{-1}(u + k),$$

or, expanding by Taylor's Theorem,

$$x + h = x + \frac{dx}{du}k + \frac{d^2x}{du^2}\frac{k^2}{1\,.\,2} + \frac{d^3x}{du^3}\frac{k^3}{1\,.\,2\,.\,3} + \&c.$$

But from the given equation we have

$u + k = 0$, or $k = -u$, and therefore

$$h = -\frac{dx}{du}u + \frac{d^2x}{du^2}\frac{u^2}{1\,.\,2} - \frac{d^3x}{du^3}\frac{u^3}{1\,.\,2\,.\,3} + \&c.$$

If we put $\dfrac{dx}{du} = -v$, then $\dfrac{d^2x}{du^2} = v\dfrac{dv}{dx}$,

$$\frac{d^3x}{du^3} = -v\frac{d}{dx}\left(v\frac{dv}{dx}\right) = -\left(v\frac{d}{dx}\right)^2 v, \text{ and so on.}$$

Hence, $h = vu + v\dfrac{d}{dx}v\,.\,\dfrac{u^2}{1\,.\,2} + \left(v\dfrac{d}{dx}\right)^2 v\,.\,\dfrac{u^3}{1\,.\,2\,.\,3} + \&c.$

This is the form of the expression which is given by Paoli, *Elementi d'Algebra*, Vol. II. p. 40.

(14) As an example of Laplace's Theorem, let us take

$$y = \log (z + x \sin y),$$

and expand ϵ^y in terms of x.

Here $f(y) = \epsilon^y$, $F(z) = \log z$, $fF(z) = f_{,}(z) = z$,

$\phi(y) = \sin y$, $\phi F(z) = \phi_{,}(z) = \sin(\log z)$.

Therefore $f_{,}'(z) = 1$, $\dfrac{d}{dz}\{\phi_{,}(z)\}^2 = 2 \sin(\log z) \cos(\log z) \,.\, \dfrac{1}{z}$

$$= \sin(2 \log z) \,.\, \frac{1}{z} = \sin(\log z^2) \,.\, \frac{1}{z}.$$

$$\frac{d^2}{dz^2}\{\phi_{,}(z)\}^3 = \frac{3\sin(\log z)}{z^2}\left[2 - 3\{\sin(\log z)\}^2 - \sin(\log z)\cos(\log z)\right]$$

$$= \frac{3\sin(\log z)}{4z^2}\{8 - 9\sin(\log z) - 2\sin(\log z^2) + 3\sin(\log z^3)\}.$$

Whence $\quad \epsilon^y = z + \sin(\log z)\,\dfrac{x}{1} + \dfrac{\sin(\log z^2)}{z}\,\dfrac{x^2}{1\,.\,2} +$

$$+ \frac{3\sin(\log z)}{4z^2}\{8 - 9\sin(\log z) - 2\sin(\log z^2) + 3\sin(\log z^3)\}\frac{x^3}{1.2.3}$$

$$+ \text{\&c.}$$

(15) Again, let $\quad y = \epsilon^{z + x\cos y}$:

expand y in terms of x.

Here $F(z) = \epsilon^z$, $\phi_{,}(z) = \cos \epsilon^z$, $F'(z) = \epsilon^z$. Therefore

$$y = \epsilon^z + \epsilon^z \cos(\epsilon^z)\,\frac{x}{1} + \epsilon^z \cos(\epsilon^z)\{\cos(\epsilon^z) - 2 \sin \epsilon^z.(\epsilon^z)\}\frac{x^2}{1\,.\,2}$$

$$+ \epsilon^z \cos(\epsilon^z)\{(\cos\epsilon^z)^2 - 9\epsilon^z\cos(\epsilon^z)\sin(\epsilon^z)$$

$$+ 9\,\epsilon^{2z}(\sin\epsilon^z)^2 - 3\epsilon^{2z}\}\frac{x^3}{1\,.\,2\,.\,3} + \text{\&c.}$$

Sec. 4. *Expansion of Functions by particular methods.*

The preceding Theorems sometimes fail from the function which is to be expanded becoming infinite or indeterminate for particular values of the variable, and, more frequently, they become inapplicable from the complication of the processes necessary for determining the successive differential coefficients. Recourse must then be had to particular artifices depending on the nature of the function which is given. One of the most useful methods is to assume a series with indeterminate coefficients, and then to compare the differential of the function with that of the assumed series; so that by equating the coefficients of like powers of the variables conditions are found for determining the assumed coefficients. This method has the advantage of furnishing the law of dependence of any coefficient on those which precede it.

Ex. (1) Let $u = \epsilon^{\epsilon^x}$. (1)

Assume

$$u = a_0 + a_1 x + a_2 x^2 + \&c. + a_n x^n + \&c. \qquad (2)$$

Differentiating (1) we have

$$\frac{du}{dx} = \epsilon^x . \epsilon^{\epsilon^x}. \qquad (3)$$

Differentiating (2) we have

$$\frac{du}{dx} = a_1 + 2a_2 x + \&c. + n a_n x^{n-1} + (n+1) a_{n+1} x^n + \&c. \qquad (4)$$

Now $\epsilon^x = 1 + x + \frac{x^2}{1.2} + \&c. + \frac{x^n}{1.2...n} + \&c.$,

and substituting in (3) for ϵ^{ϵ^x} the assumed series, it becomes

$$\frac{du}{dx} = \{a_0 + a_1 x + a_2 x^2 + \&c. + a_n x^n + \&c.\} \qquad (5)$$

$$\times \left\{1 + x + \frac{x^2}{1.2} + \&c. + \frac{x^n}{1.2...n} + \&c.\right\}.$$

Comparing now the coefficients of x^n in (4) and (5) we find

$$a_{n+1} = \frac{1}{n+1}\left\{a_n + a_{n-1} + \frac{a_{n-2}}{1.2} + \frac{a_{n-3}}{1.2.3} + \&c. + \frac{a_0}{1.2...n}\right\}:$$

whence any coefficient is determined by means of those which precede it, except the first or a_0, the value of which is easily found by putting $x = 0$ in the original equation, in which case $a_0 = \epsilon^{\epsilon^0} = \epsilon$. Therefore, forming the successive coefficients from this first one,

$$\epsilon^{\epsilon^x} = \epsilon\left\{1 + x + \frac{2x^2}{1.2} + \frac{5x^3}{1.2.3} + \frac{15x^4}{1.2.3.4} + \frac{52x^5}{1.2.3.4.5} + \&c.\right\}.$$

(2) Let $u = \epsilon^{\sin x}$.

Then by the same process as before we find the coefficient of the general term to be given by the equation

$$a_{n+1} = \frac{1}{n+1}\left\{a_n - \frac{a_{n-2}}{1.2} + \frac{a_{n-4}}{1.2.3.4} - \frac{a_{n-6}}{1.2.3.4.5.6} + \&c.\right\}.$$

There remains to be determined a_0, which is easily seen to be equal to 1. Hence we find

$$\epsilon^{\sin x} = 1 + x + \frac{x^2}{1.2} - \frac{3x^4}{1.2.3.4} - \frac{8x^5}{1.2.3.4.5} - \frac{3x^6}{1.2.3.4.5.6} + \&c.$$

(3) Let $u = \epsilon^{\sin^{-1} x}$.

Then $\dfrac{du}{dx} = \dfrac{1}{(1-x^2)^{\frac{1}{2}}}\epsilon^{\sin^{-1} x}$;

and $(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.$

Therefore, assuming a series as in the preceding examples, we find for determining the coefficient of the general term,

$$a_{n+1} = \frac{1}{n+1}\left\{a_n + \frac{1}{2}a_{n-2} + \frac{1.3}{2.4}a_{n-4} + \&c.\right\}:$$

Also, it is easily seen, that $a_0 = 1$, therefore

$$\epsilon^{\sin^{-1} x} = 1 + x + \frac{x^2}{1.2} + \frac{2x^3}{1.2.3} + \frac{5x^4}{1.2.3.4} + \frac{20x^5}{1.2.3.4.5} + \&c.$$

(4) Let $u = (a_0 + a_1 x + a_2 x^2 + \&c. + a_n x^n + \&c.)^m$.

Assume this to be equal to

$$A_0 + A_1 x + A_2 x^2 + \&c. + A_n x^n + \&c.$$

and take the logarithmic differentials of both expressions: equating these we have

$$\frac{m\left\{a_1 + 2a_2x + 3a_3x^2 + \&c. + (n+1)\,a_{n+1}x^n + \&c.\right\}}{a_0 + a_1x + a_2x^2 + \&c. + a_nx^n + \&c.}$$

$$= \frac{A_1 + 2A_2x + 3A_3x^2 + \&c. + (n+1)\,A_{n+1}x^n + \&c.}{A_0 + A_1x + A_2x^2 + \&c. + A_nx^n + \&c.}.$$

Whence, multiplying up the denominators and equating the coefficients of like powers of x, we have

$$(n+1)\,a_0A_{n+1} = (m-n)\,a_1A_n + \left\{2m-(n-1)\right\}a_2A_{n-1}$$
$$+ \left\{3m-(n-2)\right\}a_3A_{n-2} + \left\{4m-(n-3)\right\}a_4A_{n-3} + \&c.$$

Also since u is reduced to a_0^n when $x = 0$, we have $A_0 = a_0^n$. Therefore

$$u = a_0^n + ma_1a_0^{n-1}x + m\left(\frac{m-1}{2}a_1^2 + a_2a_0\right)a_0^{n-2}x^2$$
$$+ m\left\{\frac{(m-1)(m-2)}{2\,.\,3}a_1^3 + (m-1)\,a_2a_1a_0 + a_3a_0^2\right\}a_0^{n-3}x^3 + \&c.$$

Euler, *Calc. Diff.* p. 519.

(5) Let $u = \epsilon^{a_0+a_1x+a_2x^2+\dots\dots}$

As before, assume

$$u = A_0 + A_1x + A_2x^2 + \&c. + A_nx^n + \&c.$$

By taking the logarithmic differentials we find

$$(n+1)\,A_{n+1} = a_1A_n + 2a_2A_{n-1} + \&c. + (n+1)\,a_{n+1}A_0.$$

Also since $u = \epsilon^{a_0}$ when $x = 0$, $A_0 = \epsilon^{a_0}$, so that we have

$$u = \epsilon^{a_0}\left\{1 + a_1x + \frac{a_1^2 + 2a_2}{1\,.\,2}x^2 + \frac{a_1^3 + 6a_1a_2 + 6a_3}{1\,.\,2\,.\,3}x^3 + \&c.\right\}$$

Euler, *Ib.* p. 535.

In some cases the law of the coefficients is best found by proceeding to two differentiations.

(6) Let it be required to expand $\cos nx$ in ascending powers of $\cos x$.

Assume

$$\cos nx = a_0 + a_1 \cos x + \&c. \dots + a_p (\cos x)^p + \dots + a_{p+2} (\cos x)^{p+2} + \&c.$$

Differentiating,

$$n \sin nx = \{a_1 + 2a_2 \cos x + \dots + p a_p (\cos x)^{p-1} + \dots$$
$$+ (p+2)\, a_{p+2} (\cos x)^{p+1} + \&c.\} \sin x.$$

Differentiating again,

$$n^2 \cos nx = a_1 \cos x + \dots + p a_p (\cos x)^p + \dots + (p+2)\, a_{p+2} (\cos x)^{p+2}$$
$$+ \&c.$$
$$- \{2a_2 + \dots + p(p-1)\, a_p (\cos x)^{p-2} + \dots$$
$$+ (p+1)(p+2)\, a_{p+2} (\cos x)^p + \dots\} (\sin x)^2.$$

Putting $1 - (\cos x)^2$ for $(\sin x)^2$, and taking the coefficient of $(\cos x)^p$ we find it to be

$$p a_p + p(p-1)\, a_p - (p+1)(p+2)\, a_{p+2};$$

and this must be equal to the coefficient of $(\cos x)^p$ in the original series multiplied by n^2: equating these we have the condition

$$a_{p+2} = -\frac{(n^2 - p^2)}{(p+1)(p+2)}\, a_p,$$

by means of which any coefficient is given in terms of that two places below it. There remain to be determined by other means the first two coefficients a_0 and a_1. For this purpose make $x = (2r+1)\frac{\pi}{2}$ in the original equation, r being any integer. Every term on the second side vanishes except the first, and there remains

$$a_0 = \cos n (2r+1) \frac{\pi}{2}.$$

To find a_1, make $x = (2r+1)\frac{\pi}{2}$ in the second equation, when we obtain

$$a_1 = n\frac{\sin n(2r+1)\frac{\pi}{2}}{\sin(2r+1)\frac{\pi}{2}} = n\cos(n-1)(2r+1)\frac{\pi}{2}.$$

Starting from these values and giving p successively all integer values from 0 upwards, we find

$$\cos nx = \cos n(2r+1)\frac{\pi}{2}\left\{1 - \frac{n^2}{1.2}(\cos x)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\cos x)^4 - \&c.\right\}$$

$$+ \cos(n-1)(2r+1)\frac{\pi}{2}\left\{n\cos x - \frac{n(n^2-1^2)}{1.2.3}(\cos x)^3 + \&c.\right\}$$

When n is an even integer the second line, being multiplied by an odd multiple of $\frac{\pi}{2}$, vanishes, and the first line alone remains: when n is an odd integer the first line vanishes and the second line alone remains. When n is a fraction both lines must be retained, except for some particular values of n which cause the factor of one or other series to vanish.

(7) To expand $\sin nx$ in ascending powers of $\sin x$.

Proceeding in the same manner as in the last example, we find

$$\sin nx = \sin nr\pi\left\{1 - \frac{n^2}{1.2}(\sin x)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin x)^4 - \&c.\right\}$$

$$+ \cos(n-1)r\pi\left\{n\sin x - \frac{n(n^2-1^2)}{1.2.3}(\sin x)^3 + \&c.\right\}$$

When n is an integer the first series always vanishes, and the second is positive or negative according as $(n-1)r$ is even or odd. When n is odd the second series terminates; when n is even it continues to infinity. When n is fractional both series coexist, except for particular values of r.

(8) To expand $\cos nx$ in ascending powers of $\sin x$, and $\sin nx$ in ascending powers of $\cos x$.

Proceeding as in the last two examples, we find

$$\cos nx = \cos nr\pi \left\{1 - \frac{n^2}{1.2}(\sin x)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\sin x)^4 - \&c.\right\}$$

$$- \sin(n-1)r\pi \left\{n \sin x - \frac{n(n^2-1^2)}{1.2.3}(\sin x)^3 + \&c.\right\}$$

When n is an integer the second line always disappears, and the first series terminates when n is even, and does not terminate when n is odd. When n is fractional both series are retained, except for particular values of r.

$$\sin nx = \sin n(2r+1)\frac{\pi}{2}\left\{1 - \frac{n^2}{1.2}(\cos x)^2 + \frac{n^2(n^2-2^2)}{1.2.3.4}(\cos x)^4 - \&c.\right\}$$

$$+ \sin(n-1)(2r+1)\frac{\pi}{2}\left\{n\cos x - \frac{n(n^2-1^2)}{1.2.3}(\cos x)^3 + \&c.\right\}$$

When n is an odd integer the first line, when n is even the second, alone remains; but when n is fractional both series are retained except for particular values of r. In no case do the series ever terminate.

For an exposition of the difficulties concerning these expansions, and the discussions to which they have given rise, the reader is referred to Poinsot's *Memoir on Angular Sections*, where the complete form of these expansions was first given.

(9) To expand $\frac{x}{\epsilon^x - 1}$ in ascending powers of x.

If we were to endeavour to effect this by means of Maclaurin's Theorem, we should find that all the differential coefficients take the form $\frac{0}{0}$. An artifice of Laplace* however enables us to avoid this difficulty. Since

$$\frac{x}{\epsilon^x - 1} = \frac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}} - 1} - \frac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}} + 1},$$

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* *Mémoires de l'Académie*, 1777, p. 106.

the coefficient of x^n in $\dfrac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}}-1} - \dfrac{x}{\epsilon^x - 1}$ is the same as that of x^n in $\dfrac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}}+1}$. Now it is easy to shew that $\dfrac{x}{\epsilon^x - 1}$ can contain no odd powers of x above the first; for if we assume

$$\frac{x}{\epsilon^x - 1} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&c.,$$

we have $\dfrac{x}{\epsilon^{-x}-1} = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \&c.;$

and subtracting the latter from the former,

$$\frac{x(1-\epsilon^x)}{\epsilon^x - 1} = -x = 2\{a_1 x + a_3 x^3 + \&c.\},$$

and comparing the coefficients of like powers of x,

$$a_1 = -\tfrac{1}{2}, \qquad a_3 = 0, \qquad a_5 = 0, \quad \&c.$$

Also it is easy to see that $a_0 = 1$; we may therefore assume

$$\frac{x}{\epsilon^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{1.2} - B_3 \frac{x^4}{1.2.3.4} + \&c.$$

B_1, B_3, &c. being coefficients to be determined.

The coefficient of x^{2n} in the expansion of $\dfrac{x}{\epsilon^x - 1}$ is therefore $(-)^{n+1} \dfrac{B_{2n-1}}{1.2.3\ldots(2n)}$, and consequently the corresponding coefficient in

$$\frac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}} - 1} \text{ is } (-)^{n+1} \frac{1}{2^{2n}} \frac{B_{2n-1}}{1.2.3\ldots(2n)}.$$

If therefore C_{2n} be the coefficient of x^{2n} in $\dfrac{\frac{1}{2}x}{\epsilon^{\frac{x}{2}}+1}$, we have the equation

$$(-)^{n+1}\left(\frac{1}{2^{2n}}-1\right)\frac{B_{2n-1}}{1.2.3\ldots(2n)} = C_{2n}.$$

But $C_{2n} = \frac{1}{2^{2n}}\frac{1}{1.2\ldots(2n-1)}\left(\frac{d}{dx}\right)^{2n-1}\left(\frac{1}{\epsilon^x+1}\right)$, when $x = 0$,

and if in Ex. (27) of Chap. II. Sec. 1, we make $r = 2n-1$, $x = 0$,

$$\left(\frac{d}{dx}\right)^{2n-1}\left(\frac{1}{\epsilon^x+1}\right) = (-)^{2n-1}\frac{1}{2^{2n}}\left[1^{2n-1} - \left\{2^{2n-1} - \frac{2n}{1}1^{2n-1}\right\}\right.$$
$$\left. + \left\{3^{2n-1} - \frac{2n}{1}2^{2n-1} + \frac{2n(2n-1)}{1.2}1^{2n-1}\right\} - \&c.\right].$$

Substituting these values, we find

$$B_{2n-1} = \frac{(-)^{n+1}2n}{2^{2n}(2^{2n}-1)}\left[1^{2n-1} - \left\{2^{2n-1} - \frac{2n}{1}1^{2n-1}\right\}\right.$$
$$\left. + \left\{3^{2n-1} - \frac{2n}{1}2^{2n-1} + \frac{2n(2n-1)}{1.2}1^{2n-1}\right\} - \&c.\right].$$

These coefficients B_1, $B_3 \ldots B_{2n-1}$, are of great use in the expansion of series, and bear the name of Bernoulli's numbers, having been first noticed by James Bernoulli in his posthumous work the *Ars Conjectandi*, p. 97; but the complete investigation of the law of their formation is due to Euler, *Calc. Diff.* Part II, Cap. V.

(10) To expand $\tan\theta$ by means of the numbers of Bernoulli

$$\tan\theta = \frac{1}{(-)^{\frac{1}{2}}}\frac{\epsilon^{(-)^{\frac{1}{2}}2\theta}-1}{\epsilon^{(-)^{\frac{1}{2}}2\theta}+1} = \frac{1}{(-)^{\frac{1}{2}}}\left\{1 - \frac{2}{1+\epsilon^{(-)^{\frac{1}{2}}2\theta}}\right\}.$$

The coefficient of θ^{2n-1} in the expansion of this function will be the same as that of x^{2n} in the development of $\frac{x}{\epsilon^x+1}$ multiplied by $2^{2n}(-)^n$. By what has preceded it appears, therefore, to be equal to

$$\frac{2^{2n}(2^{2n}-1)}{1.2\ldots(2n)}B_{2n-1}.$$

Hence, giving n successive values, we find

$$\tan\theta = \frac{4.3}{1.2} B_1 \theta + \frac{16.15}{1.2.3.4} B_3 \theta^3 + \frac{64.63}{1.2.3.4.5.6} B_5 \theta^5 + \&\text{c.}$$

(11) To expand $\cot\theta$ by means of Bernoulli's numbers.

$$\cot\theta = (-)^{\frac{1}{2}} \frac{\epsilon^{(-)^{\frac{1}{2}} 2\theta} + 1}{\epsilon^{(-)^{\frac{1}{2}} 2\theta} - 1} = (-)^{\frac{1}{2}} \left\{1 + \frac{2}{\epsilon^{(-)^{\frac{1}{2}} 2\theta} - 1}\right\}.$$

Now the coefficient of θ^{2n-1} in this expression is the same as that of x^{2n} in the expansion $\dfrac{x}{\epsilon^x - 1}$ multiplied by $(-)^n 2^{2n}$: it is therefore equal to

$$\frac{(-)^n 2^{2n} B_{2n-1}}{1.2.3\ldots(2n)}.$$

Hence we have

$$\cot\theta = \frac{1}{\theta} - \frac{2^2}{1.2} B_1 \theta - \frac{2^4}{1.2.3.4} B_3 \theta^3 - \&\text{c.}$$

CHAPTER VI.

EVALUATION OF FUNCTIONS WHICH FOR CERTAIN VALUES OF THE VARIABLE BECOME INDETERMINATE.

IF u be a function of x of the form $\frac{P}{Q}$, and if for the value $x = a$, P and Q both vanish; u, taking the form $\frac{0}{0}$, is indeterminate and its true value will be found by differentiating the numerator and denominator separately and taking the quotient of these differentials: that is, using Lagrange's notation, the real value of u will be

$$u = \frac{P'}{Q'}, \text{ when } x = a.$$

But if the same value ($x = a$) which makes P and Q vanish also make $P' = 0$, and $Q' = 0$, we must differentiate again, and so on in succession, as long as the numerator and the denominator both vanish when x is put equal to a. Therefore we may say generally that the true value of u when $x = a$ is

$$u = \frac{P^{(n)}}{Q^{(n)}};$$

$P^{(n)}$ and $Q^{(n)}$ being the first differential coefficients of P and Q which do not vanish simultaneously when x is put equal to a.

This theory of the evaluation of indeterminate functions was first given by John Bernoulli, *Acta Eruditorum*, 1704, p. 375.

Ex. (1) $$u = \frac{1 - x^n}{1 - x} \text{ when } x = 1.$$

Here $P = 1 - x^n$; $\qquad Q = 1 - x$;

$P' = -nx^{n-1}$; $\qquad Q' = -1$;

and therefore when $x = 1$, $u = n$.

The function $\dfrac{1 - x^n}{1 - x}$ is the sum of the series

$$1 + x + x^2 + \text{ \&c. } + x^{n-1},$$

which when $x = 1$ is equal to n, as we have just found.

(2) $$u = \frac{a(ax)^{\frac{1}{2}} - x^2}{a - (ax)^{\frac{1}{2}}}, \text{ when } x = a.$$

The real value is $u = 3a$.

(3) $$u = \frac{x^3 - 1}{x^3 + 2x^2 - x - 2}, \text{ when } x = 1.$$

The real value is $u = \frac{1}{2}$.

(4) $$u = \frac{(2a^3x - x^4)^{\frac{1}{2}} - a(a^2x)^{\frac{1}{3}}}{a - (ax^3)^{\frac{1}{4}}}, \text{ when } x = a.$$

The real value is $u = \dfrac{16}{9}a$.

This was one of the first functions the value of which was determined in this manner.

(5) Find the real value of

$$u = \frac{ax^2 + ac^2 - 2acx}{bx^2 - 2bcx + bc^2}, \text{ when } x = c.$$

Differentiating twice, we find $u = \dfrac{0}{0} = \dfrac{a}{b}$.

(6) Let $$u = \frac{x^3 - ax^2 - a^2x + a^3}{x^2 - a^2};$$

when $x = a$, $u = 0$.

(7) Let $$u = \frac{ax - x^2}{a^4 - 2a^3x + 2ax^3 - x^4};$$

when $x = a$, $u = \infty$.

(8) $$u = \frac{b - (b^2 - x^2)^{\frac{1}{2}}}{x^2}, \text{ when } x = 0.$$

After two differentiations we find $u = \frac{1}{2b}$.

(9) $$u = \frac{(a^2 + ax + x^2)^{\frac{1}{2}} - (a^2 - ax + x^2)^{\frac{1}{2}}}{(a + x)^{\frac{1}{2}} - (a - x)^{\frac{1}{2}}}.$$

When $x = 0$, $u = a^{\frac{1}{2}}$.

One of the most important applications of this process is to find the sums of series for particular values of the variable. The first example was an instance of this, and we shall here add others taken like that from Euler's *Calc. Diff.* p. 746.

(10) The sum of the series

$$x + 2x^2 + 3x^3 + \ldots\ldots + nx^n,$$

is $$u = \frac{x - (n + 1)x^{n+1} + nx^{n+2}}{(1 - x)^2};$$

and we are required to find its value when $x = 1$, or when the series becomes that of the first order of figurate numbers. By two differentiations we find

$$u = \frac{0}{0} = \frac{n(n + 1)}{2}.$$

(11) The sum of the series

$$x + 4x^2 + 9x^3 + \ldots\ldots + n^2x^n,$$

is $$u = \frac{x + x^2 - (n + 1)^2 x^{n+1} + (2n^2 + 2n - 1)x^{n+2} - n^2x^{n+3}}{(1 - x)^3};$$

and we are required to find the real value of u when $x = 1$, which in this case is the sum of the squares of the natural numbers. Differentiating three times we have

$$u = \frac{0}{0} = \frac{n(n + 1)(2n + 1)}{6}.$$

(12) Let $$u = \frac{x^m - x^{m+n}}{1 - x^{2p}};$$

find its value when $x = 1$. Since u may be put under the form

$$u = \frac{x^m}{1 + x^p} \frac{1 - x^n}{1 - x^p},$$

of which the latter factor alone becomes indeterminate, we need only differentiate that factor so as to find its real value when $x = 1$, and then multiply it by the value of the first factor when $x = 1$. The real value of the fraction is

$$u = \frac{n}{2p}.$$

(13) Let $$u = \frac{a^n - x^n}{\log a - \log x}.$$

When $x = a$, $u = na^n$.

(14) Let $$u = \frac{\log x}{(1 - x)^{\frac{1}{2}}}.$$

When $x = 1$, $u = 0$.

(15) Find the true value of

$$u = \frac{\epsilon^{mx} - \epsilon^{ma}}{(x - a)^r}, \text{ when } x = a.$$

Differentiating r times with respect to x we find the real value to be

$$u = \frac{m^r \epsilon^{ma}}{r(r-1)\ldots\ldots 3 . 2 . 1}.$$

This result is of use in the integration of differential equations.

(16) Let $$u = \frac{\epsilon^x - \epsilon^{-x}}{\log(1 + x)}.$$

When $x = 0$, $u = 2$.

(17) $$u = \frac{x^x - x}{1 - x + \log x}, \text{ when } x = 1.$$

After two differentiations we find $u = -2$.

(18) Let $u = \left(\frac{\sin m\theta}{\sin \theta}\right)^2$, m being an integer.

When $\theta = 0$, $u = m^2$.

Airy's *Tracts*, p. 328.

(19) $$u = \frac{1 - \cos x}{x \log (1 + x)}, \text{ when } x = 0.$$

After two differentiations we find $u = \frac{1}{2}$.

(20) Let $u = \dfrac{\cos x\theta - \cos n\theta}{n^2 - x^2}$;

find its value when $x = n$.

$$\text{Here } P' = -\theta \sin x\theta, \quad Q' = -2x,$$

$$\text{and } u = \frac{\theta}{2n} \sin n\theta, \text{ when } x = n.$$

(21) Take the more general case of

$$u = \frac{\cos x\theta - \cos n\theta}{(n^2 - x^2)^r}, \text{ when } x = n$$

Here u may be put under the form

$$u = \frac{1}{(n + x)^r} \frac{\cos x\theta - \cos n\theta}{(n - x)^r},$$

and as the second factor alone becomes indeterminate for $x = n$, we differentiate it, and then multiply by the first factor, as in Ex. 12.

$$\text{Let then } P = \cos x\theta - \cos n\theta, \quad Q = (n - x)^r.$$

Differentiating r times with respect to x,

$$P^{(r)} = \theta^r \cos (x\theta + \tfrac{1}{2}r\pi) = \theta^r \cos (n\theta + \tfrac{1}{2}r\pi), \text{ when } x = n,$$

$$Q^{(r)} = (-)^r r (r-1)\ldots\ldots 3 . 2 . 1.$$

Also when $x = n$, $\dfrac{1}{(n+x)^r}$ becomes $\dfrac{1}{(2n)^r}$; therefore we obtain for the real value of u,

$$u = (-)^r \frac{\theta^r \cos (n\theta + \frac{1}{2} r\pi)}{r(r-1)\ldots\ldots 2.2.1\,(2n)^r}.$$

The results in the last two examples are of considerable importance in the Integral Calculus.

(22) Let $u = \dfrac{1 - \sin x + \cos x}{\sin x + \cos x - 1}$.

$$\text{When } x = \frac{\pi}{2}, \quad u = 1.$$

The method employed in the preceding examples applies only to those functions of which the differential coefficients do not become infinite when $x = a$. It will therefore fail in those cases in which the numerator and denominator of the fraction involve fractional powers of $x - a$, or generally in those functions for which Taylor's Theorem is said to fail. The following process is equally applicable in all cases, and in some of the preceding examples might perhaps be found practically more convenient. "If $x = a$ cause the fraction to assume the form $\dfrac{0}{0}$, substitute $a + h$ for x in both numerator and denominator, and develop both according to powers of h; reduce the new fraction to its simplest form, and then make $h = 0$; the result will be the true value of the fraction." It is easily seen that this includes the method by differentiation.

(23) $$u = \frac{(x^2 - a^2)^{\frac{3}{2}}}{(x-a)^{\frac{3}{2}}}, \quad \text{when } x = a.$$

In this case the first differential coefficients both become equal to 0, for $x = a$; while the second and all the subsequent differential coefficients become infinite for the same value of x.

Let $x = a + h$, then

$$u = \frac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = \frac{(2a)^{\frac{3}{2}} h^{\frac{3}{2}} \left(1 + \dfrac{h}{2a}\right)^{\frac{3}{2}}}{h^{\frac{3}{2}}};$$

and expanding the binomial,

$$u = \frac{(2a)^{\frac{3}{2}} h^{\frac{3}{2}} \left(1 + \frac{3}{2}\,\frac{h}{2a} + \&c.\right)}{h^{\frac{3}{2}}} :$$

$$u = (2a)^{\frac{3}{2}} \left(1 + \frac{3}{2}\,\frac{h}{2a} + \&c.\right)$$

$$= (2a)^{\frac{3}{2}}, \quad \text{when } h = 0;$$

and this is the true value of the fraction.

(24) $$u = \frac{(a^2 - x^2)^{\frac{1}{2}} + a - x}{(a-x)^{\frac{1}{2}} + (a^3 - x^3)^{\frac{1}{2}}} = \frac{0}{0}, \quad \text{when } x = a.$$

Since in this case u would be impossible if x were greater than a, assume $x = a - h$; the result is

$$u = \frac{(2a)^{\frac{1}{2}}}{1 + 3^{\frac{1}{2}} a}, \quad \text{when } h = 0,$$

which is the real value of the fraction.

(25) $$u = \frac{\tan \pi x - \pi x}{2x^2 \tan \pi x}, \quad \text{when } x = 0.$$

Let $x = 0 + h$, or h; then, expanding the circular function by the formula

$$\tan\theta = \theta + \frac{2\theta^3}{1.2.3} + \&c.$$

we have $$u = \frac{1}{2h^2} - \frac{\pi}{2h\left\{\pi h + \frac{2(\pi h)^3}{1.2.3} + \&c.\right\}}$$

$$= \frac{\pi + \frac{2\pi^3 h^2}{1.2.3} + \&c. - \pi}{2h\left\{\pi h + \frac{2(\pi h)^3}{1.2.3} + \&c.\right\}} = \frac{\pi^2}{6}, \quad \text{when } h = 0.$$

This is the sum of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \text{\&c. to } \infty.$$

There are other forms of functions which become indeterminate for a particular value of the variable, but these may generally be reduced to the form $\frac{0}{0}$. Thus, if $u = P \, . \, Q$, and if for $x = a$, $P = 0$, $Q = \infty$, we have $u = 0 \times \infty$, which is indeterminate. But if we assume $Q = \frac{1}{Q_1}$, we have

$$u = \frac{P}{Q_1} = \frac{0}{0}, \quad \text{when } x = a.$$

Therefore, applying the rule for vanishing fractions,

$$u = \frac{P'}{Q_1'} = -Q^2 \frac{P'}{Q'}, \quad \text{when } x = a.$$

But if $Q = \infty$ when $x = a$, all its differential coefficients will also be infinite, and u, taking the form $\frac{\infty}{\infty}$, is still indeterminate, unless the factor which becomes infinite should happen to divide out.

To determine the value of a function $u = \frac{P}{Q}$, which becomes $\frac{\infty}{\infty}$ for $x = a$, we assume $P = \frac{1}{P_1}$, $Q = \frac{1}{Q_1}$, so that $u = \frac{Q_1}{P_1} = \frac{0}{0}$, when $x = 0$.

Treating this as a vanishing fraction,

$$u = \frac{Q_1'}{P_1'} = \frac{P^2}{Q^2} \frac{Q'}{P'}, \quad \text{when } x = a.$$

$$\text{Whence } \quad u = \frac{P}{Q} = \frac{P'}{Q'}, \quad \text{when } x = a.$$

But if $P = \infty$, and $Q = \infty$, for $x = a$, all their differential coefficients will also become infinite, and $\frac{P'}{Q'}$ will still have the form $\frac{\infty}{\infty}$. This reduction, then, will not give us the true value of the fraction unless some factor divide out, or we can find some relation, depending on the nature of the functions, between the new numerator and denominator which will enable us to trace the real value.

A function $u = P - Q$, which becomes $\infty - \infty$ when $x = a$, can frequently be reduced to the form $\frac{0}{0}$; for if $P = \frac{1}{P_1}$, and $Q = \frac{1}{Q_1}$,

$$u = \frac{Q_1 - P_1}{P_1 Q_1} = \frac{0}{0}, \quad \text{when } x = a,$$

and its value is to be found by the usual method.

De Morgan's *Differential Calculus*, p. 172.

(26) $$u = (1 - x) \tan \frac{\pi}{2} x = 0 . \infty, \quad \text{when } x = 1.$$

The real value is $u = \frac{2}{\pi}$, when $x = 1$.

(27) $$u = \epsilon^{\frac{1}{x}} \sin x, \quad \text{when } x = 0;$$

$$u = 0 . \infty = \infty.$$

(28) $$u = \epsilon^{-\frac{1}{x}} (1 - \log x), \quad \text{when } x = 0;$$

Here $P' = -\frac{1}{x}$, $Q' = -\frac{1}{x^2} \epsilon^{\frac{1}{x}}$,

and $u = \epsilon^{-\frac{1}{x}} . x = 0$, when $x = 0$.

(29) $$u = \frac{x^n}{\epsilon^x} = \frac{\infty}{\infty}, \quad \text{when } x = \infty.$$

Differentiating the numerator and denominator n times in succession,

$$u = \frac{n (n - 1) \ldots 2 . 1}{\epsilon^x} = 0, \quad \text{when } x = \infty.$$

(30) $$u = \frac{\log x}{x^n}, \text{ when } x = \infty:$$

$$u = \frac{P'}{Q'} = \frac{1}{nx^n} = 0, \text{ when } x = \infty.$$

(31) $u = x^n \log x$ when $x = 0$, n being positive.

Put $x = \frac{1}{y}$. Then $u = -\frac{\log y}{y^n}$, and when $x = 0$, $y = \infty$; and therefore by the last example

$$u = x^n \log x = 0, \text{ when } x = 0.$$

If n be negative, $u = -\infty$.

(32) $$u = \frac{1}{1-x} - \frac{2}{1-x^2} = \infty - \infty, \text{ when } x = 1.$$

$$u = \frac{1+x-2}{1-x^2} = \frac{0}{0} = -\tfrac{1}{2}, \text{ when } x = 1.$$

(33) $$u = \frac{x}{x-1} - \frac{1}{\log x} = \infty - \infty, \text{ when } x = 1:$$

$$u = \frac{x \log x - x + 1}{(x-1)\log x} = \frac{0}{0} = \tfrac{1}{2}, \text{ when } x = 1.$$

(34) The sum of the series

$$\frac{1}{1+x^2} + \frac{1}{4+x^2} + \frac{1}{9+x^2} + \&\text{c. to } \infty,$$

$$\text{is } u = \frac{\pi x - 1}{2x^2} + \frac{\pi}{x(\epsilon^{2\pi x} - 1)};$$

find its value when $x = 0$.

Here $$u = \frac{(\pi x - 1)(\epsilon^{2\pi x} - 1) + 2\pi x}{2x^2(\epsilon^{2\pi x} - 1)} = \frac{0}{0}, \text{ when } x = 0.$$

Differentiating three times, we find the real value to be

$$u = \frac{\pi^2}{6},$$

which is the sum of the reciprocals of the squares of the natural numbers.

(35) The sum of the series

$$\frac{1}{1^2+x^2}+\frac{1}{3^2+x^2}+\frac{1}{5^2+x^2}+\text{\&c. to }\infty,$$

is $u=\dfrac{\pi}{4x}-\dfrac{\pi}{2x(\epsilon^{\pi x}+1)}=\infty-\infty$, when $x=0$:

$$u=\frac{\pi}{4}\,\frac{\epsilon^{\pi x}-1}{x(\epsilon^{\pi x}+1)}=\frac{0}{0},\ \text{ when } x=0.$$

Differentiating once, we find

$$u=\frac{\pi^2}{8},$$

which is therefore the sum of the squares of the reciprocals of the odd numbers.

The reader will find other examples of a similar kind relating to the summation of series in Euler's *Calc. Diff.* p. 760, seq.

Sometimes the value of an indeterminate function may be most readily found by throwing it into a form in which its real nature is more easily seen.

(36) If $\qquad u=2^x\sin\dfrac{a}{2^x}$;

find its value when $x=\infty$.

This function may be put under the form $a\dfrac{\sin\frac{a}{2^x}}{\frac{a}{2^x}}$,

which, if $\dfrac{a}{2^x}=\theta$, becomes $a\dfrac{\sin\theta}{\theta}$. When $x=\infty$, $\theta=0$, and $\dfrac{\sin\theta}{\theta}=1$, and therefore $u=a$.

(37) Let $u=(a^2-x^2)^{\frac{1}{2}}\cot\left\{\dfrac{\pi}{2}\left(\dfrac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}=0\,.\,\infty$, when $x=a$.

This may be put under the form

$$u = \frac{(a^2 - x^2)^{\frac{1}{2}}}{\tan\left\{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}\right\}} = \frac{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}}{\tan\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}} \times \frac{2}{\pi}(a+x);$$

$$\text{and as } \frac{\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}}{\tan\frac{\pi}{2}\left(\frac{a-x}{a+x}\right)^{\frac{1}{2}}} = 1, \text{ when } x = a,$$

$$\text{we have } u = \frac{4a}{\pi}, \text{ when } x = a.$$

(38) Find the value of $u = \frac{\log(\tan 2x)}{\log(\tan x)}$, when $x = 0$.

$$\text{Log}(\tan 2x) = \log\left(\frac{2\tan x}{1 - \tan^2 x}\right) = \log(2\tan x) - \log(1 - \tan^2 x);$$

$$\text{therefore } u = \frac{\log(\tan x) + \log 2 - \log(1 - \tan^2 x)}{\log(\tan x)}$$

$$u = 1 + \frac{\log 2 - \log(1 - \tan^2 x)}{\log\tan x} = 1, \text{ when } x = 0.$$

Functions which for a particular value of the variable take the form 0^0 ∞^0 $1^{\pm\infty}$, may be reduced to a shape in which the preceding methods are applicable. Let z and y be functions of x and $u = z^y$, then if for $x = a$

$z = 0$, $y = 0$ we have $u = 0^0$,

$z = \infty$, $y = 0$ we have $u = \infty^0$,

$z = 1$, $y = \pm\infty$ we have $u = 1^{\pm\infty}$.

Now since $z = \epsilon^{\log z}$, $u = \epsilon^{y\log z}$; and these three cases are reduced to the determination of $y \log z$, which takes the form $0 \times \pm\infty$.

De Morgan's *Diff. Calc.* p. 175.

(39) Find the value of x^{x^a} when $x = 0$, a being positive. This is equivalent to $\epsilon^{x^a \log x}$, and we have to find the value of $x^a \log x$ when $x = 0$.

Now by Ex. (31) $x^a \log x = 0$ when $x = 0$. Therefore

$$x^{x^a} = \epsilon^0 = 1 \text{ when } x = 0.$$

If a be negative $x^{x^a} = 0$ when $x = 0$.

(40) Find the value of

$$u = \left(\frac{1}{x^n}\right)^{x^m} = \infty^0 \text{ when } x = 0.$$

$$u = \epsilon^{-x^m \log (x^n)} = \epsilon^{-n x^m \log x}.$$

As before, $x^m \log x = 0$ when $x = 0$;

$$\text{therefore} \quad u = \epsilon^0 = 1 \text{ when } x = 0.$$

(41) $u = x^{\sin x}$ when $x = 0$.

We may arrive at the value of this function by the consideration that, when x is indefinitely diminished, $\dfrac{\sin x}{x} = 1$, or $\sin x = x$: therefore when $x = 0$, $x^{\sin x} = x^x = 1$, by Ex. (39).

In the same way it would appear that

$$(\sin x)^{\sin x} = 1 \text{ when } x = 0.$$

Also, since $x^{\sin x} = \epsilon^{\log x \,.\, \sin x} = 1$ when $x = 0$,

it appears that $\quad \sin x \,.\, \log x = 0$ when $x = 0$;

and similarly that $\quad \sin x \,.\, \log (\sin x) = 0$ when $x = 0$.

(42) $u = (\cot x)^{\sin x} = \infty^0$ when $x = 0$.

By similar considerations it appears that

$$(\cot x)^{\sin x} = 1 \text{ when } x = 0.$$

(43) $u = (1 + nx)^{\frac{1}{x}}$ when $x = 0$.

Here $u = \epsilon^{\frac{\log(1+nx)}{x}}$, and we have to find the value of $\frac{\log(1+nx)}{x}$ when $x = 0$. Differentiating we find this to be $\frac{n}{1+nx} = n$ when $x = 0$. Therefore

$$u = (1+nx)^{\frac{1}{x}} = \epsilon^n \text{ when } x = 0.$$

This result may be verified by expanding u by the binomial theorem: that gives

$$u = 1 + \frac{1}{x}(nx) + \frac{1}{x}\left(\frac{1}{x} - 1\right)\frac{n^2x^2}{1.2} + \frac{1}{x}\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)\frac{n^3x^3}{1.2.3} + \&c.$$

$$= 1 + n + \frac{n^2}{1.2}(1 - x) + \frac{n^3}{1.2.3}(1 - 3x + 2x^2) + \&c.$$

$$= 1 + n + \frac{n^2}{1.2} + \frac{n^3}{1.2.3} + \&c. = \epsilon^n \text{ when } x = 0.$$

(44) $u = (\cos ax)^{(\operatorname{cosec}\beta x)^2}$ when $x = 0$;

The real value is $u = \epsilon^{\frac{a^2}{2\beta^2}}$.

Functions which for a particular value of the variable take the form 0^0, have been used by Libri to introduce discontinuity into ordinary functions.

Thus, if it be desired to express a function $f(x)$ which shall be equal to $\phi(x)$ from $x = -\infty$ to $x = n$, and to $\psi(x)$ from $x = n$ to $x = \infty$, he writes

$$f(x) = (1 - 0^{0^{x-n}})\,\phi(x) + 0^{0^{x-n}}\,\psi(x).$$

See his *Mémoires de Mathématique et de Physique*, Vol. I. p. 44, and Crelle's *Journal*, Vol. x. p. 303. In the same *Journal*, Vol. XII. p. 134 and p. 292, the reader will find some discussion on the real value of this indeterminate expression.

CHAPTER VII.

MAXIMA AND MINIMA.

SECT. 1. *Explicit Functions of One Variable.*

IF u be an explicit function of x, then the real roots of $\frac{du}{dx} = 0$ are the values of x which render u a maximum or a minimum.

If a root of $\frac{du}{dx} = 0$ substituted in $\frac{d^2u}{dx^2}$ give a negative result, the corresponding value of u is a *maximum*. If a root of $\frac{du}{dx} = 0$ substituted in $\frac{d^2u}{dx^2}$ give a positive result, the corresponding value of u is a *minimum*.

If a root of $\frac{du}{dx} = 0$ also make $\frac{d^2u}{dx^2} = 0$, the corresponding value of u is neither a maximum nor a minimum, unless $\frac{d^3u}{dx^3}$ also vanish for the same value; and generally, u is neither a maximum nor a minimum for a given value of x, unless the first differential coefficient which does not vanish for that value of x be of an even order.

If, when $x + h$ is substituted for x in u, the expansion in terms of h assume the form Ph^{α} + &c., where α is less than unity, the values of x which make u a maximum or minimum are to be sought in the roots of the equation

$$\frac{1}{\frac{du}{dx}} = 0.$$

If u be of the form $\frac{P}{Q}$, the equation for determining x is $Q\frac{dP}{dx} - P\frac{dQ}{dx} = 0$, unless the same value of x make $Q = 0$, in which case $\frac{du}{dx} = \frac{0}{0}$, and its real value must be found by the usual method. The corresponding value of u will be a maximum or a minimum according as the value of x when substituted in $Q\frac{d^2P}{dx^2} - P\frac{d^2Q}{dx^2}$ gives a negative or a positive result. The values of x which make $Q = 0$ make $u = \infty$, unless $P = 0$ at the same time; and this value of u will sometimes be a maximum, sometimes not. If the same value of x which makes $u = \infty$ make $\frac{1}{u}$ a minimum, then $u = \infty$ is a maximum;—otherwise not.

The investigation of the maximum and minimum values of u is sometimes facilitated by the following considerations.

If u be a maximum or minimum, and a be a positive quantity, au is also a maximum or minimum.

When u is a maximum or minimum, au^{2n+1} is so also; but $\frac{a}{u^{2n+1}}$ is inversely a minimum or maximum.

If u be a positive maximum or minimum, au^{2n} is also a maximum or minimum. If u be a negative maximum or minimum, au^{2n} will be a minimum or maximum. The same remarks apply to fractional powers of the function u, except that when the denominator of the fraction is even and the value of u negative, the power of u is impossible.

When u is a maximum or minimum, $\log u$ is generally a maximum or minimum, except when the same value of the variable makes both u and $\frac{du}{dx}$ vanish. This preparation of the function is frequently made when the function u consists of products or quotients of roots and powers, as the differentiation is thus facilitated.

Other transformations of u are sometimes useful, but as these depend on particular forms which but rarely occur, they may be left to the ingenuity of the student who desires to simplify the solution of the proposed problem.

In testing by the sign of $\frac{d^2u}{dx^2}$ whether the value of u be a maximum or a minimum, the following consideration will sometimes shorten the process.

If $\frac{du}{dx}$ be of the form $v_1 . v_2 . v_3 \ldots v_n$, and $x = a$ be a value which causes one of the factors as v_r to vanish, the only term of $\frac{d^2u}{dx^2}$ which is to be considered is that involving $\frac{dv_r}{dx}$, as all the others vanish with v_r when x is put equal to a, so that $\frac{d^2u}{dx^2}$ is reduced to one term.

Ex. (1) $$u = x - x^2.$$

$$\frac{du}{dx} = 1 - 2x = 0, \quad \text{whence} \quad x = \tfrac{1}{2};$$

$$\frac{d^2u}{dx^2} = -2, \text{ and } x = \tfrac{1}{2} \text{ makes } u = \tfrac{1}{4}, \text{ a maximum.}$$

(2) $$u = x^4 - 8x^3 + 22x^2 - 24x + 12.$$

$$\frac{du}{dx} = 4x^3 - 24x^2 + 44x - 24 = 0,$$

$$\text{or} \quad x^3 - 6x^2 + 11x - 6 = 0.$$

The roots of this equation are 1, 2, 3, and

for $x = 1$, u is a minimum;
for $x = 2$, u is a maximum;
for $x = 3$, u is a minimum.

(3) $$u = x^5 - 5x^4 + 5x^3 + 1.$$

$$\frac{du}{dx} = 5x^4 - 20x^3 + 15x^2 = 0,$$

$$\text{or} \qquad x^4 - 4x^3 + 3x^2 = 0,$$

the roots of which are 0, 0, 1, 3.

The roots 0 make u neither a maximum nor a minimum;

the root 1 makes u a maximum;

the root 3 makes u a minimum.

(4) $$u = (x - a)^n.$$

$$\frac{du}{dx} = n(x - a)^{n-1};$$

$x = a$ makes $u = 0$, which is a minimum when n is even, and neither a maximum nor a minimum when n is odd.

(5) $$u = x^m (a - x)^n.$$

$$\frac{du}{dx} = x^{m-1}(a - x)^{n-1}\{ma - (m + n)x\} = 0;$$

the roots of which are $x = 0$, $x = a$, and $x = \dfrac{ma}{m+n}$;

$x = 0$ makes $u = 0$, a minimum if m be even, and neither a maximum nor a minimum if m be odd.

$x = a$ makes $u = 0$, a minimum if n be even, and neither a maximum nor a minimum if n be odd.

$$\frac{d^2u}{dx^2} = \frac{d}{dx}\{x^{m-1}(a-x)^{n-1}\}\{ma-(m+n)x\} - (m+n)x^{m-1}(a-x)^{n-1},$$

which for $x = \dfrac{ma}{m+n}$ becomes

$$-(m + n)\left(\frac{ma}{m+n}\right)^{m-1}\left(\frac{na}{m+n}\right)^{n-1};$$

and therefore $x = \dfrac{ma}{m+n}$ makes u a maximum.

This is the solution of the problem. To divide the number a into two parts, such that the product of the m^{th} power of the one by the n^{th} power of the other shall be a maximum.

(6) $$u = \frac{x}{1+x^2}; \qquad \frac{du}{dx} = \frac{1-x^2}{(1+x^2)^2} = 0.$$

Whence $x^2 = 1, \quad x = +1, \quad x = -1$;

$$\frac{d^2u}{dx^2} = -\frac{6x+2x^3}{(1+x^2)^3}.$$

If $x = +1, \quad \frac{d^2u}{dx^2} = -1, \quad u = \frac{1}{2}$, a maximum;

$x = -1, \quad \frac{d^2u}{dx^2} = 1, \quad u = -\frac{1}{2}$, a minimum.

(7) $$u = \frac{x^2-x+1}{x^2+x-1}.$$

$x = 0, \quad u = -1$, a maximum;

$x = 2, \quad u = \frac{3}{5}$, a minimum.

(8) $$u = \frac{(x+3)^3}{(x+2)^2}.$$

for $x = 0, \quad u = \frac{27}{4}$, a minimum;

for $x = -3$, u is neither a maximum nor a minimum;

for $x = -2$, $u = \infty$, a maximum.

(9) $$u = \frac{(x-1)^2}{(x+1)^3}.$$

$x = 5$ gives $u = \frac{2}{27}$, a maximum;

$x = 1$ gives $u = 0$, a minimum;

$u = -1$ gives $u = \infty$, which is neither a maximum nor a minimum.

(10) $$u = \frac{x^2}{(a^2+x^2)^3}; \qquad \frac{du}{dx} = \frac{2x(a^2-2x^2)}{(a^2+x^2)^4} = 0;$$

whence $x = \frac{a}{2^{\frac{1}{2}}}$, and $u = \frac{4}{27a^4}$, a maximum.

This is the solution of the problem, To find the height at which a light should be placed so that a small plane surface at a given horizontal distance shall receive the greatest illumination from it.

(11) $$u = \frac{x}{(a+x)(b+x)}.$$

$$x = (ab)^{\frac{1}{2}}, \quad u = \frac{1}{(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2}, \text{ a maximum.}$$

This is a solution of the problem, To find the magnitude of the body which must be interposed between two others so that the velocity communicated from the one to the other shall be a maximum.

(12) $$u = m \sin (x - a) \cos x.$$

$$x = \frac{a}{2} + \frac{\pi}{4} \text{ gives } u = \frac{m}{2}(1 - \sin a), \text{ a maximum;}$$

$$x = \frac{a}{2} - \frac{\pi}{4} \text{ gives } u = -\frac{m}{2}(1 + \sin a), \text{ a minimum.}$$

This is a solution of the problem, To find in what direction a body must be projected with a given velocity that its range on a given plane may be the greatest possible.

(13) $$u = \frac{(\sin mx)^2}{(\sin x)^2}, \; m \text{ being an integer.}$$

The values of x derived from $m \tan x = \tan mx$, make u a maximum.

The values of x derived from $\sin mx = 0$, make u a minimum.

The values of x derived from $\sin x = 0$, make u a maximum and equal to m^2.

Airy's *Tracts*, p. 328.

(14) $$u = \frac{\log x}{x^n}.$$

$$x = \epsilon^{\frac{1}{n}} \text{ gives } u = \frac{1}{n\epsilon}, \text{ a maximum.}$$

(15) $u = x^{\frac{1}{x}}$; $x = \epsilon$ gives $u = \epsilon^{\frac{1}{\epsilon}}$, a maximum.

(16)
$$u = (a^2 + c^2 - 2cx)^{\frac{1}{2}} + x.$$

$$x = \frac{a^2}{2c} \text{ gives } u = \frac{a^2 + 2c^2}{2c}, \text{ a maximum.}$$

(17)
$$u = \frac{(a^2 x - x^3)^{\frac{1}{2}}}{2^{\frac{1}{2}} a^{\frac{3}{2}} + x^{\frac{3}{2}}}.$$

$$x = \frac{a}{2} \text{ gives } u = \frac{1}{3^{\frac{3}{2}}}, \text{ a maximum.}$$

(18) Let $u = b + c(x - a)^{\frac{2}{3}}$.

Here $\dfrac{du}{dx} = \frac{2}{3} c (x - a)^{-\frac{1}{3}}$,

and $\dfrac{1}{\frac{du}{dx}} = 0$, when $x = a$.

If $x = a + h$, $u_1 = b + ch^{\frac{2}{3}}$;
$x = a - h$, $u_2 = b + ch^{\frac{2}{3}}$.

Therefore if c be positive $x = a$ makes u a minimum; and if c be negative, a maximum.

(19) Let $u = b + c(x - a)^{\frac{5}{3}}$.

It will be seen by the same means as in the last example, that $x = a$ gives neither a maximum nor a minimum.

(20)
$$u = (1 + x^{\frac{2}{3}})(7 - x)^2.$$

$x = 7$ gives u a minimum;
$x = 1$ gives u a maximum;
$x = 0$ gives u a minimum.

(21) To divide a number a into a number of equal parts such that their continued product shall be a maximum.

Let x be the number of parts, then $\dfrac{a}{x}$ is one of the parts, and $\left(\dfrac{a}{x}\right)^x$ is the continued product, which is to be a maximum.

Taking the logarithmic differential, we have

$$\log a - \log x - 1 = 0.$$

Whence $\log x = \log a - 1 = \log a - \log \epsilon = \log \frac{a}{\epsilon}$;

therefore $x = \frac{a}{\epsilon}$, and $\left(\frac{a}{x}\right)^x = \epsilon^{\frac{a}{\epsilon}}$.

(22) From two points A, B (fig. 1), to draw two straight lines to a point P in a given line ON such that $AP + BP$ shall be a minimum.

Take the given line as the axis of x, O the origin.

Let $OP = x$, and let the co-ordinates of A and B be a, b, a_1, b_1. Then

$$u = AP + BP = \{b^2 + (x-a)^2\}^{\frac{1}{2}} + \{b_1{}^2 + (a_1 - x)^2\}^{\frac{1}{2}} = \text{minimum.}$$

Whence $\frac{x-a}{\{b^2 + (x-a)^2\}^{\frac{1}{2}}} = \frac{a_1 - x}{\{b_1{}^2 + (a_1 - x)^2\}^{\frac{1}{2}}}$;

or the angles APM, BPN are equal.

(23) To find the point in the straight line AD (fig. 2), at which BC subtends the greatest angle; ABC being perpendicular to AD.

If the angle be a maximum its tangent is also a maximum.

Let P be the point, $AP = x$, $AC = a$, $AB = b$;

$$\tan BPC = \tan (APC - APB) = \frac{(a-b)\,x}{ab + x^2};$$

therefore $\frac{x}{ab + x^2}$ is to be a maximum.

Whence $ab - x^2 = 0$ and $x = (ab)^{\frac{1}{2}}$.

(24) Through a point M (fig. 3) within the angle BAC draw the line PQ so that the triangle PAQ shall be a minimum.

Draw MN parallel to AQ, and let $AN = a$, $MN = b$, $AP = x$. Then $x = 2a$ makes PAQ a minimum, and PQ is bisected in M.

(25) Given the length of the arc of a circle, find the angle which it must subtend at the centre in order that the corresponding segment may be a maximum.

Let a be the half-length of the arc, x the radius of the circle; then $\frac{a}{x}$ is the half-angle of the segment.

If u be the segment,

$$u = ax - x^2 \sin \frac{a}{x} \cos \frac{a}{x} \text{ is a maximum.}$$

$$\text{Whence} \quad \cos \frac{a}{x} \left(a \cos \frac{a}{x} - x \sin \frac{a}{x} \right) = 0.$$

If $\cos \frac{a}{x} = 0$, or $\frac{a}{x} = \frac{\pi}{2}$, the segment is a semicircle and a maximum.

If $\frac{a}{x} = \tan \frac{a}{x}$, $x = \infty$, and $u = 0$ is a minimum.

This last equation might be equally satisfied analytically by a value of $\frac{a}{x}$ between π and $\frac{3\pi}{2}$, but such an angle is excluded by the geometry of the problem.

A geometrical solution of this problem is given in the *Mathematical Collections* of Pappus, Book V. Theor. 16.

(26) AC (fig. 4) and BD being parallel, it is required to draw from C a line CXY, such that the sum of the triangles ACX and BXY shall be a minimum.

If $AC = a$, $AB = b$, $AX = x$, it is easily seen that the area of the triangle ACX is proportional to ax, and that of BXY to $\frac{a(b-x)^2}{x}$, so that we have

$$a \left\{ x + \frac{(b-x)^2}{x} \right\}, \text{ a minimum.}$$

Whence we find $x^2 - b^2 = 0$, or $x = b^{\frac{1}{2}}$, which determines the line CXY.

Vincent Viviani, *Geometrica Divinatio*, p. 152.

(27) *OM*, *OP* (fig. 5) are two arcs of great circles on a sphere, and the arc *PM* is drawn perpendicular to *OM*; find when the difference between *OP* and *OM* is the greatest.

Let $POM = \alpha$, $OP = \phi$, $OM = \theta$; then we have

$$\phi - \theta = \text{a maximum};$$

while, by Napier's rules for the solution of right-angled spherical triangles,

$$\tan\theta = \cos\alpha \tan\phi.$$

Differentiating with respect to θ, we have

$$\frac{d\phi}{d\theta} - 1 = 0,$$

$$1 + \tan^2\theta = \cos\alpha\,(1 + \tan^2\phi)\frac{d\phi}{d\theta};$$

therefore $1 + \tan^2\theta = \cos\alpha\,(1 + \tan^2\phi)$,

or $1 + \cos^2\alpha \tan^2\phi = \cos\alpha\,(1 + \tan^2\phi)$.

Whence $\tan\phi = (\sec\alpha)^{\frac{1}{2}}$, $\tan\theta = (\cos\alpha)^{\frac{1}{2}}$.

(28) To determine the dimensions of a cylinder open at the top, which, under the least surface, shall contain a given volume.

Let πa^3 be the given volume, x the radius of the cylinder, y its base. Then $x = y = a$ determines the cylinder of least surface.

(29) The content of a cone being given, find its form when the surface is a maximum.

Let $\dfrac{\pi a^3}{3}$ be the given content, x the radius of the base, y the height of the cone. Then $x = \dfrac{a}{2^{\frac{1}{2}}}$, $y = 2a$ determine the cone of greatest surface.

(30) To inscribe the greatest cone in a given sphere.

Let the radius of the sphere be r, and the distance of the base of the cone from the centre be x. Then $x = \frac{1}{3}r$ gives the maximum cone.

(31) To find the point in the line joining the centres of two spheres, from which the greatest portion of spherical surface is visible.

If r, r' be the radii of the spheres, a the distance of the centres, and x the distance of the required point from the centre of the sphere whose radius is r.

Then $$x = \frac{ar^{\frac{3}{2}}}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}}.$$

(32) A regular hexagonal prism is regularly terminated by a trihedral solid angle formed by planes each passing through two angles of the prism; find the inclination of these planes to the axis of the prism in order that for a given content the total surface may be the least possible.

Let $ABCabc$ (fig. 6) be the base of the prism, $PQRS$ one of the faces of the terminating solid angle passing through the angles P, R. Let S be the vertex of the pyramid. Draw SO perpendicular to the upper surface of the prism. Join OM, RP, SQ intersecting each other in N. Then it is easy to see that $MN = NO$ and consequently $SO = QN$, and as the triangles POR, PMR are equal, the pyramids $PSRO$ and $PMRQ$ are equal, so that, whatever be the inclination of SQ to OM, the part cut off from the prism is equal to the part included in the pyramid SPR, and the content of the whole therefore remains constant. We have then to determine the angle ONS or OSN so that the total surface shall be a minimum. Let AB, the side of the hexagon, $= a$, AP, the height of the prism, $= b$, $OSN = \theta$. Then

$ON = NM = \frac{1}{2}a$, and $SN = \frac{1}{2}a \operatorname{cosec}\theta$, and $QM = \frac{1}{2}a\cot\theta$.

The surface of $ABPQ = \frac{1}{2}a(2b - \frac{1}{2}a\cot\theta)$.

The surface of $PQRS = PR \,.\, SN = \dfrac{3^{\frac{1}{2}}a^2}{2}\operatorname{cosec}\theta$.

Whence the total surface of the solid is

$$3a\left(2b - \frac{a}{2}\cot\theta\right) + \frac{3^{\frac{3}{2}}a^2}{2}\operatorname{cosec}\theta,$$

which is to be a minimum. Differentiating we have

$$\frac{1}{\sin^2\theta} - 3^{\frac{1}{2}}\frac{\cos\theta}{\sin^2\theta} = 0; \quad \text{and therefore } \cos\theta = \frac{1}{3^{\frac{1}{2}}}.$$

Hence $\tan SRN = \dfrac{1}{2^{\frac{1}{2}}}$, and $\tan SRQ = 2^{\frac{1}{2}}$.

This is the celebrated problem of the form of the cells of bees. Maraldi* was the first who measured the angles of the faces of the terminating solid angle, and he found them to be $109^{0}.28'$ and $70^{0}.32''$ respectively. It occurred to Réaumur that this might be the form, which, for the same solid content, gives the minimum of surface, and he requested Kœnig to examine the question mathematically. That geometer confirmed the conjecture;—the result of his calculations agreeing with Maraldi's measurements within $2'$. Maclaurin† and L'Huillier‡, by different methods, verified the preceding result, excepting that they shewed that the difference of $2'$ was due to an error in the calculations of Kœnig—not to a mistake on the part of the bees.

(33) To determine the greatest parabola which can be cut from a given cone.

Let ABC (fig. 7) be the cone $BC = a$, $AC = b$, $BN = x$.

Then $DN = \dfrac{b}{a} x$, and $EN = (ax - x^2)^{\frac{1}{2}}$.

The area of the parabola is $\dfrac{4}{3} EN . DN$, or

$$\frac{4}{3}\frac{b}{a} x (ax - x^2)^{\frac{1}{2}}, \text{ which is to be a maximum.}$$

Whence $x = \dfrac{3a}{4}$, and the area of the parabola $= \dfrac{ab\,3^{\frac{1}{2}}}{4}$.

(34) To determine the greatest ellipse which can be cut from a given cone.

Let AC (fig. 8) $= a$, $CD = b$, $CN = x$, BP being the major axis of the ellipse. Then the condition that the area of the ellipse shall be a maximum gives

$$x = \frac{2b(a^2 - b^2) \pm b(a^4 - 14a^2b^2 + b^4)^{\frac{1}{2}}}{3(a^2 + b^2)}.$$

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* *Mémoires de l'Académie des Sciences*, 1712, p. 299.

† *Philosophical Transactions*, 1743, p. 565.

‡ *Berlin Memoirs*, 1781, p. 277.

In order that this may be possible we must have

$$a^4 - 14a^2b^2 + b^4 > 0, \text{ or } a > b\left\{2 + 3^{\frac{1}{2}}\right\};$$

that is, the angle of the cone must be less than 15°.

When this is not the case, the ellipse increases continually till it coincides with the base.

It may happen that the maximum value of the section is less than the base of the cone; and this will be the case unless the vertical angle of the cone be less than 11°. 57′.

SECT. 2. *Implicit Functions of Two Variables.*

If u be an implicit function of two variables x and y, $\dfrac{du}{dx} = 0$ will determine the values of x for which y is a maximum or minimum. y will be a maximum for a given value of x when that value substituted in $\dfrac{\dfrac{d^2u}{dx^2}}{\dfrac{du}{dy}}$ gives a positive result, and a minimum when it gives a negative result.

Ex. (1) $$u = y^2 - x^2y + x - x^3 = 0.$$

$$\frac{du}{dx} = 1 - 2xy - 3x^2 = 0;$$

whence $y = \dfrac{1 - 3x^2}{2x}$, and substituting this value, x is determined from the equation

$$2x^5 + 9x^4 + 2x^3 - 6x^2 + 1 = 0:$$

one root of this equation is $x = -1$, which gives $y = 1$; and as $\dfrac{\dfrac{d^2u}{dx^2}}{\dfrac{du}{dy}} = \dfrac{-6x - 2y}{2y - x^2} = +4$, $y = 1$ is a maximum.

(2) $$y^2 + 2yx^2 + 4x - 3 = 0.$$

$x = -\frac{1}{2}$ gives $y = 2$, a maximum;

$x = 1$ gives $y = -1$, neither a maximum nor a minimum.

(3) $y^3 + x^3 - 3a^2x = 0.$

$x = +a$ gives $y = 2^{\frac{1}{3}}a$, a maximum;

$x = -a$ gives $y = -2^{\frac{1}{3}}a$, a minimum.

(4) $u = y^3 + x^3 - 3axy = 0.$

$x = 2^{\frac{1}{3}}a$ gives $y = 4^{\frac{1}{3}}a$, a maximum;

$x = 0$ gives $y = 0$, a minimum.

The nature of this last value cannot be determined by the usual method, since $\frac{dy}{dx} = \frac{0}{0}$, and also $\frac{d^2y}{dx^2} = \frac{0}{0}$. To determine it, differentiate u a second time considering y and x both to vary, y being a function of x; we then get, on making $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$, or $= \frac{a}{0}$. Differentiating a third time on the same supposition, and making $x = 0$, $y = 0$, $\frac{dy}{dx} = 0$, we find $\frac{d^2y}{dx^2} = \frac{2}{3a}$, which being positive shows that $y = 0$ is a minimum.

(5) $u = y^4 + x^4 - 4xy + 2.$

$\frac{dy}{dx} = \frac{y - x^3}{y^3 - x}$, whence $y = x^3$;

and $x^{12} - 3x^4 + 2 = 0$;

whence $x = +1$ or -1, and $y = +1$ or -1.

Therefore $\frac{dy}{dx} = \frac{0}{0}$; and proceeding as in the last example we find that $y = +1$ and $y = -1$ are neither maxima nor minima.

(6) $u = y^2 - 2mxy + x^2 - a^2 = 0.$

$$x = \frac{ma}{(1 - m^2)^{\frac{1}{2}}} \text{ gives } y = \frac{a}{(1 - m^2)^{\frac{1}{2}}};$$

and as these values of x and y when substituted in $\frac{d^2y}{dx^2}$ give a negative result, the value of y is a maximum.

The preceding examples are taken from Euler's *Calc. Diff.* Part II. Cap. XI.

SECT. 3. *Functions of two or more Variables.*

Let u be a function of two variables x and y; then the conditions for u being a maximum or minimum, are

$$\frac{du}{dx} = 0; \qquad \frac{du}{dy} = 0.$$

In addition to these Lagrange has shewn* that the condition $\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} > \left(\frac{d^2u}{dx\,dy}\right)^2$ must also hold, in order that the function may not change its character from a maximum to a minimum, or *vice versâ*, in the course of the changes of the variables. From this condition it appears that $\frac{d^2u}{dx^2}$ and $\frac{d^2u}{dy^2}$ must have the same sign; and the function will be a maximum if that sign be negative, and a minimum if it be positive.

If the values of x and y which make $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$, also make the second differentials vanish, there will be no maximum or minimum unless the third differentials also vanish, while the fourth neither vanish nor change their sign.

If u be a function of three variables $x\ y\ z$, the conditions for a maximum or minimum are

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0;$$

and Lagrange's condition becomes in this case

$$\left\{\frac{d^2u}{dx^2}\,\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2\right\}\left\{\frac{d^2u}{dx^2}\,\frac{d^2u}{dz^2} - \left(\frac{d^2u}{dx\,dz}\right)^2\right\} >$$
$$\left(\frac{d^2u}{dy\,dz} \cdot \frac{d^2u}{dx^2} - \frac{d^2u}{dx\,dy}\,\frac{d^2u}{dx\,dz}\right)^2.$$

In like manner if there be a function of n variables x, y, z, t... we shall have, for determining their values when the function is a maximum or minimum, the n equations:

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0, \quad \frac{du}{dt} = 0, \text{ \&c.}$$

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* *Turin Memoirs*, Vol. I. p. 19.

In this case Lagrange's condition becomes too complicated to be easily expressed, and as such functions rarely if ever occur in practice, it is unnecessary to give it here.

Français has shewn* that in a function of two variables, when the equations

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0,$$

are satisfied simultaneously by the vanishing of one factor, they are really equivalent to only one condition, by which we cannot determine x and y, but can only find a relation between them. This corresponds geometrically to a *locus* of maxima and minima, such as would be produced by the extremity of the major axis of an ellipse which revolves round an axis parallel to the major axis. In these cases we have

$$\frac{d^2u}{dx^2} \cdot \frac{d^2u}{dy^2} - \left(\frac{d^2u}{dx\,dy}\right)^2 = 0,$$

an equation which is usually excluded from Lagrange's condition. It is to be observed, however, that this view supposes a *maximum* to be a value *not less* than any other immediately contiguous, whereas it is generally considered to be a value *greater* than any other, and conversely for a minimum. This remark of Français is of more importance geometrically than analytically; and I may add, that in geometry the failure of Lagrange's condition indicates that there is a maximum for some sections, and a minimum for others.

When a function of two or more variables is to be made a maximum or minimum, it frequently happens that there are given certain equations of condition between the variables, so that the real number of *independent* variables is less than the number of variables in the function. When this is the case, instead of getting rid of the superfluous variables by direct elimination, it is usually more convenient to employ Lagrange's method of indeterminate multipliers.† The following is the theory of the method.

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* *Annales de Gergonne*, Vol. III. p. 132.

† *Mécanique Analytique*, Vol. I. p. 74.

Let u be a function of n variables $x_1, x_2, \dots x_n$, these being subject to the r equations of condition,

$$L_1 = 0, \quad L_2 = 0, \dots\dots L_r = 0.$$

When u is a maximum or minimum,

$$du = \frac{du}{dx_1} dx_1 + \frac{du}{dx_2} dx_2 + \dots + \frac{du}{dx_n} dx_n = 0 \qquad (1),$$

which is to be combined with the r equations

$$dL_1 = 0, \quad dL_2 = 0, \dots dL_r = 0 \qquad (2),$$

the general form of each of which is

$$\frac{dL}{dx_1} dx_1 + \frac{dL}{dx_2} dx_2 + \dots + \frac{dL}{dx_n} dx_n = 0.$$

These equations are all linear in $dx_1, dx_2, \dots dx_n$; multiply therefore the equations (2) by indeterminate multipliers $\lambda_1, \lambda_2, \dots \lambda_r$, and add them to (1). We then get

$$du + \lambda_1 dL_1 + \lambda_2 dL_2 + \dots + \lambda_r dL_r = 0;$$

an equation which is of the form

$$M_1 dx_1 + M_2 dx_2 + \dots + M_n dx_n = 0,$$

in which each quantity M is of the form

$$M = \frac{du}{dx} + \lambda_1 \frac{dL_1}{dx} + \lambda_2 \frac{dL_2}{dx} + \dots + \lambda_r \frac{dL_r}{dx}.$$

If we determine the r quantities λ from the conditions that they make the terms involving $dx_1, dx_2, \dots dx_r$ vanish, that is to say, if we determine them by the conditions

$$M_1 = 0, \quad M_2 = 0, \dots M_r = 0,$$

the variations $dx_1, dx_2, \dots dx_r$ are eliminated, and there remains

$$M_{r+1} dx_{r+1} + \dots + M_n dx_n = 0:$$

and as the $n-r$ quantities $dx_{r+1} \dots dx_n$ are independent, their coefficients are separately equal to zero. Hence if we determine $\lambda_1, \lambda_2, \dots \lambda_r$ by the equations

$$M_1 = 0, \quad M_2 = 0 \dots M_r = 0,$$

we shall also have

$$M_{r+1} = 0 \dots M_n = 0;$$

and the n quantities x and the r quantities λ satisfy the $n+r$ equations,

$$M_1 = 0, \quad M_2 = 0 \dots M_n = 0, \quad L_1 = 0 \dots L_r = 0.$$

As it is indifferent which of the variables we eliminate in order to determine $\lambda_1 \dots \lambda_r$, the most general way of stating the result is, that we have the $n+r$ equations,

$$L_1 = 0, \quad L_2 = 0 \dots L_r = 0,$$

$$\left.\begin{array}{l} \dfrac{du}{dx_1} + \lambda_1 \dfrac{dL_1}{dx_1} + \lambda_2 \dfrac{dL_2}{dx_1} + \dots + \lambda_r \dfrac{dL_r}{dx_1} = 0. \\ \dotfill \\ \dfrac{du}{dx_n} + \lambda_1 \dfrac{dL_1}{dx_n} + \lambda_2 \dfrac{dL_2}{dx_n} + \dots + \lambda_n \dfrac{dL_r}{dx_n} \end{array}\right\} \quad (3),$$

to determine the $n+r$ quantities $x_1, x_2, \dots x_n, \lambda_1, \lambda_2, \dots \lambda_r$.

If u be homogeneous in $x_1, x_2, \dots x_n$, and if $L_1, L_2, \dots L_r$ consist of homogeneous terms and constants, there exists a simple relation between the quantities λ, which is very useful in many problems.

Let u be homogeneous of m dimensions, and let $L_1 = 0$, $L_2 = 0$, &c. be put under the form

$$M_a + A = 0, \quad N_b + B = 0, \text{ \&c.}$$

where M_a is homogeneous of a dimensions, N_b homogeneous of b dimensions, &c., and where A, B, &c. are constants. Then multiplying equations (3) by $x_1, x_2, \dots x_n$ and adding, we have, by a property of homogeneous functions,

$$mu + a\lambda_1 M_a + b\lambda_2 N_b + \text{\&c.} = 0,$$

$$\text{or} \quad mu = a\lambda_1 A + b\lambda_2 B + \text{\&c.}$$

Ex. (1) $\quad u = x^3 + y^3 - 3axy.$

$$\frac{du}{dx} = 3(x^2 - ay) = 0, \quad \frac{du}{dy} = 3(y^2 - ax) = 0,$$

$$\text{whence} \quad x^4 - a^3 x = 0.$$

Therefore $x = 0$, or $x = a$, and $y = 0$, $y = a$.

Lagrange's condition becomes $36xy - 9a^2 > 0$.

Therefore $x = 0$, $y = 0$ gives neither a maximum nor a minimum, and $x = a$, $y = a$ gives $u = -a^3$, a minimum when a is positive, and a maximum when a is negative.

(2) $$u = x^4 + y^4 - 2x^2 + 4xy - 2y^2,$$

$$\frac{du}{dx} = x^3 - x + y = 0, \quad \frac{du}{dy} = y^3 - y + x = 0.$$

Eliminating y between these equations, we find

$$x^3 \left\{(x^2 - 1)^3 - 1\right\} = 0,$$

whence $x = 0$, $x = \pm 2^{\frac{1}{2}}$, $y = 0$, $y = \mp 2^{\frac{1}{2}}$:

$x = 0$, $y = 0$ give $u = 0$, a maximum;

$x = \pm 2^{\frac{1}{2}}$, $y = \mp 2^{\frac{1}{2}}$ give $u = -8$, a minimum.

(3) $$u = x^3 y^2 (a - x - y),$$

$x = \frac{a}{2}$, $y = \frac{a}{3}$ give $u = \frac{a^6}{432}$ a maximum,

$x = 0$, $y = 0$, give $u = 0$ neither a maximum nor a minimum.

(4) $$u = \frac{xyz}{(a + x)(x + y)(y + z)(z + b)}.$$

Taking the logarithmic differential, since $\log u$ is a maximum when u is so. Then

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x} - \frac{1}{a + x} - \frac{1}{x + y} = 0,$$

$$\frac{1}{u}\frac{du}{dy} = \frac{1}{y} - \frac{1}{x + y} - \frac{1}{y + z} = 0,$$

$$\frac{1}{u}\frac{du}{dz} = \frac{1}{z} - \frac{1}{y + z} - \frac{1}{z + b} = 0,$$

or $ay - x^2 = 0$, $xz - y^2 = 0$, $by - z^2 = 0$;

whence $$\frac{a}{x} = \frac{x}{y} = \frac{y}{z} = \frac{z}{b},$$

or a, x, y, z, b are in geometric progression.

Let each of these ratios be equal to $\frac{1}{n}$. Then, multiplying them together, $\frac{a}{b} = \frac{1}{n^4}$, or $n = \left(\frac{b}{a}\right)^{\frac{1}{4}}$.

Let $\log u = v$, then proceeding to the second differentials we get, on substituting for x, y, z the values na, n^2a, n^3a,

$$\frac{d^2v}{dx^2} = -\frac{2}{a^2n(1+n)^2}, \quad \frac{d^2v}{dy^2} = -\frac{2}{a^2n^3(1+n)^2},$$

$$\frac{d^2v}{dz^2} = -\frac{2}{a^2n^5(1+n)^2}, \quad \frac{d^2v}{dx\,dy} = \frac{1}{a^2n^2(1+n)^2},$$

$$\frac{d^2v}{dy\,dz} = \frac{1}{a^2n^4(1+n)^2}, \quad \frac{d^2v}{dx\,dz} = 0;$$

so that Lagrange's condition becomes

$$\frac{12}{a^8n^{10}(1+n)^4} > \frac{4}{a^8n^{10}(1+n)^4},$$

and the corresponding value $u = \dfrac{1}{(a^{\frac{1}{4}} + b^{\frac{1}{4}})^4}$ is a maximum.

(5) Let $\dfrac{k}{\rho} = rx^2 + 2xy + ty^2$;

x and y being connected by the equation

$$1 = (1+p^2)x^2 + 2pq\,xy + (1+q^2)y^2.$$

Differentiating these two equations,

$$0 = (rx + sy)\,dx + (sx + ty)\,dy,$$

$$0 = \{(1+p^2)\,x + pqy\}\,dx + \{pqx + (1+q^2)\,y\}\,dy.$$

Multiply the second of these equations by an indeterminate quantity λ, add it to the first and equate to zero the coefficients of the differentials: this gives

$$\lambda\,\{(1+p^2)\,x + pqy\} + rx + sy = 0,$$

$$\lambda\,\{pqx + (1+q^2)\,y\} + sx + ty = 0.$$

Multiply these equations by x, y respectively and add, then, by the original equations $\lambda + \dfrac{k}{\rho} = 0$.

Substituting this value of λ in the preceding equations, and grouping together the terms multiplied by the same variable,

$$\left\{\frac{k}{\rho}(1+p^2) - r\right\} x = -\left(\frac{k}{\rho}pq - s\right) y,$$

$$\left\{\frac{k}{\rho}(1+q^2) - t\right\} y = -\left(\frac{k}{\rho}pq - s\right) x.$$

Multiplying these together so as to eliminate x and y, we find

$$\frac{k^2}{\rho^2}(1+p^2+q^2) - \frac{k}{\rho}\left\{(1+q^2)r - 2pqs + (1+p^2)t\right\} + rt - s^2 = 0,$$

a quadratic equation in $\frac{k}{\rho}$, whence a maximum and a minimum value may be found.

This is the equation for determining the radii of maximum and minimum curvature in a curved surface.

(6) Let $u = ay(c-z) = bz(a-x) = cx(b-y)$.

Then $x = \frac{1}{2}a$, $y = \frac{1}{2}b$, $z = \frac{1}{2}c$, give

$u = \frac{1}{4}abc$, a maximum.

(7) Let $u = a\cos^2 x + b\cos^2 y$;

x and y being subject to the condition $y - x = \frac{1}{4}\pi$.

Differentiating, we have

$$0 = a\cos x\sin x + b\cos y\sin y,$$

$$\text{or}\quad 0 = a\sin 2x + b\sin 2y.$$

From the equation of condition

$$\sin 2y = \sin\left(\frac{\pi}{2} + 2x\right) = \cos 2x.$$

Therefore $\tan 2x = -\frac{b}{a}$; whence

$$\cos^2 x = \frac{(a^2+b^2)^{\frac{1}{2}} \pm a}{2(a^2+b^2)^{\frac{1}{2}}},\quad \cos^2 y = \frac{(a^2+b^2)^{\frac{1}{2}} \pm b}{2(a^2+b^2)^{\frac{1}{2}}},$$

and $u = \frac{1}{2}\left\{a + b \pm (a^2+b^2)^{\frac{1}{2}}\right\}$.

The upper sign giving a maximum and the lower a minimum.

(8) Let $u = \cos x\cos y\cos z$,

with the condition $x + y + z = \pi$.

By taking the logarithmic differential, and using an indeterminate multiplier, we easily find

$$x = y = z = \tfrac{1}{3}\pi, \quad \text{and} \quad u = \tfrac{1}{8}, \text{ a maximum.}$$

(9) Find the maximum value of

$$u = al + bm + cn,$$

l, m, n being variable and subject to the condition

$$l^2 + m^2 + n^2 = 1.$$

We easily find by the use of an indeterminate multiplier that

$$\frac{a}{l} = \frac{b}{m} = \frac{c}{n}, \text{ and therefore}$$

$$l = \frac{a}{(a^2+b^2+c^2)^{\frac{1}{2}}}, \quad m = \frac{b}{(a^2+b^2+c^2)^{\frac{1}{2}}}, \quad n = \frac{c}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$

and therefore $u = (a^2 + b^2 + c^2)^{\frac{1}{2}}$.

This is the solution of the problem, "To find the position of the plane on which the sum of the projections of any number of planes is a maximum:" l, m, n are here the cosines of the angles which the plane of projection makes with the co-ordinate planes.

(10) Find the maximum value of

$$u = (x + 1)(y + 1)(z + 1),$$

x, y, z being subject to the condition,

$$a^x b^y c^z = A.$$

Taking the logarithmic differential of both equations we have

$$\frac{dx}{x+1} + \frac{dy}{y+1} + \frac{dz}{z+1} = 0,$$

$$dx \log a + dy \log b + dz \log c = 0.$$

Whence, by using an indeterminate multiplier λ,

$$\frac{\lambda}{x+1} = \log a, \quad \frac{\lambda}{y+1} = \log b, \quad \frac{\lambda}{z+1} = \log c.$$

From these we find

$$x+1=\frac{\log(Aabc)}{3\log a},\quad y+1=\frac{\log(Aabc)}{3\log b},\quad z+1=\frac{\log(Aabc)}{3\log c},$$

and $$u=\frac{\{\log(Aabc)\}^3}{\log a^3 . \log b^3 . \log c^3}.$$

This is the solution of the problem: If a, b, c be the prime factors of a number A, to find how many times each factor must enter into it, that it may have the greatest number of divisors. Waring, *Medit. Algeb.* p. 344.

(11) To find the rectangular parallelopiped which shall contain a given volume under the least surface.

If x, y, z be the edges of the parallelopiped, and if a^3 be the volume of a cube to which it is equal, then by the condition of the minimum we easily find

$$x=y=z=a,$$

so that the surface equals $6a^2$, a minimum.

(12) To inscribe the greatest rectangular parallelopiped in a given ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

and let x, y, z be the half edges of the parallelopiped, then

$$u=8xyz \text{ is to be a maximum,}$$

x, y, z being subject to the condition,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

By the method of indeterminate multipliers, we easily find

$$x=\frac{a}{3^{\frac{1}{2}}},\quad y=\frac{b}{3^{\frac{1}{2}}},\quad z=\frac{c}{3^{\frac{1}{2}}},\quad u=\frac{8abc}{3^{\frac{3}{2}}}.$$

(13) To find the triangle of least perimeter which can be inscribed in a given triangle.

Let ABC (fig. 9) be the given triangle, a, b, c the sides, A, B, C the angles, and DEF, being the inscribed triangle, let $CD = x$, $AE = y$, $BF = z$. Then if u be the perimeter,

$$\begin{aligned} u = {} & \{y^2 + (c - z)^2 - 2y(c - z)\cos A\}^{\frac{1}{2}} \\ & + \{x^2 + (b - y)^2 - 2x(b - y)\cos C\}^{\frac{1}{2}} \\ & + \{z^2 + (a - x)^2 - 2z(a - x)\cos B\}^{\frac{1}{2}} \end{aligned}$$

is to be a minimum. Whence we find

$$\frac{x - (b - y)\cos C}{\{x^2 + (b - y)^2 - 2x(b - y)\cos C\}^{\frac{1}{2}}} = \frac{(a - x) - z\cos B}{\{z^2 + (a - x)^2 - 2z(a - x)\cos B\}^{\frac{1}{2}}},$$

$$\frac{y - (c - z)\cos A}{\{y^2 + (c - z)^2 - 2y(c - z)\cos A\}^{\frac{1}{2}}} = \frac{(b - y) - x\cos C}{\{x^2 + (b - y)^2 - 2x(b - y)\cos C\}^{\frac{1}{2}}},$$

$$\frac{z - (a - x)\cos B}{\{z^2 + (a - x)^2 - 2z(a - x)\cos B\}^{\frac{1}{2}}} = \frac{(c - z) - y\cos A}{\{y^2 + (c - z)^2 - 2y(c - z)\cos A\}^{\frac{1}{2}}}.$$

From these equations it appears that

$$FEA = DEC, \quad EDC = BDF \quad \text{and} \quad BFD = AFE.$$

It is shown by Geometry that if lines be drawn joining the points where the perpendiculars from the angles meet the sides, each intersecting pair makes equal angles with the side in which they meet; consequently the triangle formed by these lines is the triangle of least perimeter which can be inscribed in the given triangle. See *Cambridge Mathematical Journal*, Vol. I. p. 157.

(14) To find the least distance between two straight lines in space.

Let the equations to the lines be

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = r, \tag{1}$$

$$\frac{x'-a'}{l'} = \frac{y'-b'}{m'} = \frac{z'-c'}{n'} = r'. \tag{2}$$

Then if x, y, z, x', y', z' be the co-ordinates of the extremities of the least distance (δ),

$$\delta^2 = (x-x')^2 + (y-y')^2 + (z-z')^2$$

is to be a minimum, the variables being subject to the conditions (1) and (2). Differentiating,

$$0 = (x-x')(dx - dx') + (y-y')(dy - dy') + (z-z')(dz - dz').$$

But from (1) and (2) we have

$$dx = l dr, \quad dy = m dr, \quad dz = n dr,$$
$$dx' = l' dr', \quad dy' = m' dr', \quad dz' = n' dr'.$$

Therefore, substituting these values, and, as r and r' are independent, equating to zero the coefficients of the differentials, we have the two conditions

$$l(x-x') + m(y-y') + n(z-z') = 0, \tag{3}$$

$$l'(x-x') + m'(y-y') + n'(z-z') = 0. \tag{4}$$

Between which, eliminating successively each of the quantities $(x-x')$ &c., we find

$$\frac{x-x'}{mn'-m'n} = \frac{y-y'}{nl'-n'l} = \frac{z-z'}{lm'-l'm}; \tag{5}$$

each of these ratios being equal to

$$\frac{\{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}}{\{(mn'-m'n)^2 + (nl'-n'l)^2 + (lm'-l'm)^2\}^{\frac{1}{2}}}. \tag{6}$$

Now multiply each term of (3) and (4) by the corresponding members of (1) and (2), subtract the one result from the other and transpose; then observing that

$$(x-x')^2 + (y-y')^2 + (z-z')^2 = \delta^2,$$

$$\delta^2 = (a-a')(x-x') + (b-b')(y-y') + (c-c')(z-z').$$

Dividing the first member of this by (6), and each term of the second member by each member of (5), we find

$$\delta = \frac{(a-a')(mn'-m'n)+(b-b')(nl'-n'l)+(c-c')(lm'-l'm)}{\{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2\}^{\frac{1}{2}}}.$$

Equations (5) are the equations to the line of least distance, and it appears that it is perpendicular to both the lines (1) and (2), since we have

$$l(mn'-m'n)+m(nl'-n'l)+n(lm'-l'm)=0,$$

and $l'(mn'-m'n)+m'(nl'-n'l)+n'(lm'-l'm)=0,$

which are the conditions of perpendicularity.

(15) To find the maximum and minimum radii of a section of the surface, the equation to which is

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2$$

made by the plane $lx+my+nz=0$.

We have here to find

$$r^2 = x^2+y^2+z^2, \text{ a maximum,}$$

x, y, z, being connected by the equations

$$r^4 = a^2x^2+b^2y^2+c^2z^2,$$

$$0 = lx+my+nz.$$

Differentiating, we have

$$xdx+ydy+zdz=0. \qquad (1)$$

$$a^2xdx+b^2ydy+c^2zdz=0. \qquad (2)$$

$$ldx+mdy+ndz=0. \qquad (3)$$

(1) + λ (3) − μ (2) = 0 gives, on equating to zero the coefficients of each differential,

$$x+\lambda l=\mu a^2x, \quad y+\lambda m=\mu b^2y, \quad z+\lambda n=\mu c^2z.$$

Multiply by x, y, z, and add, then by the original conditions $r^2=\mu r^4$, or $\mu=\frac{1}{r^2}$.

Substituting this value, and transposing,

$$\left(\frac{a^2}{r^2} - 1\right) x = \lambda l, \quad \left(\frac{b^2}{r^2} - 1\right) y = \lambda m, \quad \left(\frac{c^2}{r^2} - 1\right) z = \lambda n.$$

Whence $x = \dfrac{\lambda l r^2}{r^2 - a^2}, \quad y = \dfrac{\lambda m r^2}{r^2 - b^2}, \quad z = \dfrac{\lambda n r^2}{r^2 - c^2}.$

Multiply by l, m, n, and add. Then by the original conditions and dividing by λr^2,

$$\frac{l^2}{r^2 - a^2} + \frac{m^2}{r^2 - b^2} + \frac{n^2}{r^2 - c^2} = 0,$$

a quadratic equation for determining r^2, and consequently r.

This is the equation in the Wave Theory of light by which the velocities of a wave propagated in a crystalline medium are determined. The surface $r^4 = a^2x^2 + b^2y^2 + c^2z^2$ is called the surface of elasticity. See Fresnel *Mémoires de l'Institut*, Vol. VII. p. 130, and Herschel's *Light*, Sec. 1012.

(16) To find the area of a section of the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

made by the plane $lx + my + nz = 0$.

By the same method as in the last example we obtain as the equation for determining the principal axes,

$$\frac{a^2l^2}{r^2 - a^2} + \frac{b^2m^2}{r^2 - b^2} + \frac{c^2n^2}{r^2 - a^2} = 0.$$

The last term of this when arranged according to powers of r^2 is

$$\frac{a^2b^2c^2}{a^2l^2 + b^2m^2 + c^2n^2};$$

and this being equal to the product of the roots, the area of the section is

$$\frac{\pi abc}{(a^2l^2 + b^2m^2 + c^2n^2)^{\frac{1}{2}}}.$$

(17) To find the volume of the ellipsoid whose equation is

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy = c.$$

As in the preceding examples we have first to find the value of the principal axes, or rather of their product; and if this be $\alpha\beta\gamma$, then the volume of the ellipsoid will be $\frac{4\pi}{3}\alpha\beta\gamma$.

Now the principal axes are maxima or minima values of the radius; we therefore have

$$r^2 = x^2 + y^2 + z^2 \text{ a maximum;}$$

x, y, z being subject to the equation of condition

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'xz + 2b''xy = c.$$

Differentiating,

$$xdx + ydy + zdz = 0, \qquad (1)$$

$$(ax+b'z+b''y)dx+(a'y+bz+b''x)dy+(a''z+by+b'x)dz=0, \quad (2)$$

$\lambda(1) + (2) = 0$ gives, on equating to zero the coefficients of each differential,

$$\left.\begin{aligned} \lambda x + ax + b'z + b''y &= 0 \\ \lambda y + a'y + bz + b''x &= 0 \\ \lambda z + a''z + by + b'x &= 0 \end{aligned}\right\}. \qquad (3)$$

Multiply these equations by x, y, z respectively and add, then by the equation of condition,

$$\lambda r^2 + c = 0, \quad \text{and } \lambda = -\frac{c}{r^2}.$$

On substituting this value of λ in the equations (3), they become

$$\left(\frac{c}{r^2} - a\right)x - b''y - b'z = 0,$$

$$b''x - \left(\frac{c}{r^2} - a'\right)y + bz = 0,$$

$$b'x + by - \left(\frac{c}{r^2} - a''\right)z = 0.$$

To eliminate x, y, z, multiply the first of these by $\left(\frac{c}{r^2} - a'\right)\left(\frac{c}{r^2} - a''\right) - b^2$; the second by $-\left\{bb' + b''\left(\frac{c}{r^2} - a''\right)\right\}$;

the third by $-\left\{bb'' + b'\left(\frac{c}{r^2} - a'\right)\right\}$ and add, then y and z disappear, and x dividing out, there remains

$$\left(\frac{c}{r^2} - a\right)\left(\frac{c}{r^2} - a'\right)\left(\frac{c}{r^2} - a''\right) - b^2\left(\frac{c}{r^2} - a\right) - b'^2\left(\frac{c}{r^2} - a'\right)$$

$$-b''^2\left(\frac{c}{r^2} - a''\right) - 2bb'b'' = 0;$$

a cubic equation in $\frac{1}{r^2}$. If it be arranged according to powers of r^2, the last term with its sign changed will be equal to the product of the roots, that is, to the product of the squares of the principal axes; and its square root is the quantity which we seek. Multiplying it therefore by $\frac{4\pi}{3}$, we find that the volume of the ellipsoid is equal to

$$\frac{4\pi}{3}\,\frac{c^{\frac{3}{2}}}{(aa'a'' - ab^2 - a'b'^2 - a''b''^2 + 2bb'b'')^{\frac{1}{2}}}.$$

(18) To find the least ellipse which will circumscribe a given triangle.

Let ABC (fig. 10) be the triangle. Take C as the origin, CA, CB as the axes of x and y. $AC = a$, $BC = b$, $ACB = \theta$.

The general equation to an ellipse is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + 1 = 0,$$

which involves five arbitrary constants; three of these may be determined by the conditions that the ellipse shall pass through the three points A, B, C. Instead however of directly expressing the undetermined coefficients in terms of those which are determined, it will conduce to the symmetry of our analysis to assume two indeterminate quantities of which the coefficients of the equation are functions which can be determined by the conditions of the ellipse passing through the three given points; and then the actual values of the indeterminate quantities may be found by the condition of the minimum. The two quantities which we shall assume are the co-ordinates of the centre of the curve. Let them be α, β; then the equation to the ellipse may be put under the form

$$A(x-\alpha)^2 + 2B(x-\alpha)(y-\beta) + C(y-\beta)^2 + 1 = 0,$$

where A, B, C are to be determined.

Now the condition that the ellipse shall pass through the origin gives

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 + 1 = 0. \qquad (1)$$

The condition that the curve shall pass through the point $x = a$, $y = 0$, gives

$$A(a-\alpha)^2 - 2B(a-\alpha)\beta + C\beta^2 + 1 = 0. \qquad (2)$$

Subtracting (1) from (2) we have

$$A(2\alpha - a) + 2B\beta = 0. \qquad (3)$$

The condition that the curve shall pass through the point $x = 0$, $y = b$, gives

$$C(b-\beta)^2 - 2B\alpha(b-\beta) + A\alpha^2 + 1 = 0. \qquad (4)$$

Subtracting (1) from (4) we have

$$C(2\beta - b) + 2B\alpha = 0. \qquad (5)$$

Also $\alpha(3) + \beta(5) - 2(1) = 0$, gives

$$Aa\alpha + Cb\beta + 2 = 0. \qquad (6)$$

Combining (3), (5) and (6), we find

$$A = \frac{-(2\beta - b)}{\alpha(\alpha b + \beta a - ab)}, \qquad C = \frac{-(2\alpha - a)}{\beta(\alpha b + \beta a - ab)},$$

$$B = \frac{(2\alpha - a)(2\beta - b)}{2\alpha\beta(\alpha b + \beta a - ab)}.$$

It remains now to express the area of the ellipse in terms of these coefficients. The method to be adopted is the same as that used in the preceding example.

If r be any radius measured from the centre, so that

$$r^2 = (x-\alpha)^2 + (y-\beta)^2 - 2(x-\alpha)(y-\beta)\cos\theta,$$

the axes of the ellipse are determined by the equation

$$(AC - B^2)r^4 - (A + C - 2B\cos\theta)r^2 + \sin^2\theta = 0.$$

The area of the ellipse will therefore be

$$\frac{\pi \sin \theta}{(AC - B^2)^{\frac{1}{2}}},$$

which is to be a minimum, involving the condition that

$$AC - B^2 \text{ shall be a maximum.}$$

Substituting in this the values of A, B, C previously determined, and differentiating with respect to a and β, we obtain equations for determining these quantities. The result involves several factors, but those which correspond to the problem are

$$2ba + a\beta - ab = 0, \quad 2a\beta + ba - ab = 0;$$

whence $\alpha = \frac{a}{3}$, $\beta = \frac{b}{3}$, and therefore

$$A = -\frac{3}{a^2}, \quad C = -\frac{3}{b^2}, \quad B = -\frac{3}{2ab}.$$

The area of the ellipse is therefore

$$\frac{2\pi}{3^{\frac{3}{2}}} ab \sin \theta.$$

It appears then that the area of the ellipse is to that of the triangle as $4\pi : 3^{\frac{3}{2}}$, and that its centre coincides with the centre of gravity of the triangle. This problem is given by Euler in the *Nova Acta Petrop.* Vol. IX. p. 147, but his method of solution is deficient in symmetry. In the same volume he has also discussed the more general problem—To describe the least ellipse which passes through four given points.

The preceding solution is due to Bérard. *Annales de Gergonne*, Vol. IV. p. 288.

(19) To inscribe the greatest ellipse in a given triangle.

By following a method similar to that adopted in the last example it will be found that the area of the maximum ellipse is to that of the triangle as $\pi : 3^{\frac{3}{2}}$; that its centre coincides with the centre of gravity of the triangle; and that the points of contact bisect the sides of the triangle. Bérard, *Ib.* p. 284.

(20) To find a point within a triangle from which if lines be drawn to the angular points the sum of their squares is the least possible.

The centre of gravity of the triangle is the point which possesses this property.

(21) Among all triangular pyramids of given base and altitude, find that which has the least surface.

Let a, b, c be the sides of the base, h the height, θ, ϕ, ψ the angles of inclination of the faces to the base: then

$$\frac{a}{\sin\theta} + \frac{b}{\sin\phi} + \frac{c}{\sin\psi} = \text{min.}$$

θ, ϕ, ψ being subject to the condition

$$a \cot\theta + b \cot\phi + c \cot\psi = \text{const.}$$

We find that $\theta = \phi = \psi$, or that the faces are equally inclined to the base.

(22) To find a point within a triangle from which if lines be drawn to the angular points their sum may be the least possible.

The direct solution of this problem is long and complicated, but we may without much difficulty obtain a geometrical condition by which the point is readily determined.

Let ABC (fig. 11) be the given triangle, a, b, c its sides; let O be the required point, $OA = u$, $OB = v$, $OC = w$. Draw ON perpendicular to AB, and let $AN = x$, $ON = y$; also let $AON = \theta$, $BON = \phi$, $CON = \psi$. Then

$$u^2 = x^2 + y^2,\ v^2 = (c - x)^2 + y^2,\ w^2 = (b \cos A - x)^2 + (b \sin A - y)^2.$$

In order that $u + v + w$ may be a minimum, we must have

$$\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0,$$

$$\frac{du}{dy} + \frac{dv}{dy} + \frac{dw}{dy} = 0.$$

Now

$$\frac{du}{dx} = \frac{x}{u} = \sin\theta,\ \frac{dv}{dx} = -\frac{c-x}{v} = -\sin\phi,\ \frac{dw}{dx} = -\frac{b\cos A - x}{w} = -\sin\psi,$$

$$\frac{du}{dy} = \frac{y}{u} = \cos\theta,\ \frac{dv}{dy} = \frac{y}{v} = \cos\phi,\qquad \frac{dw}{dy} = -\frac{b\cos A - y}{w} = \cos\psi.$$

Therefore we have the conditions

$$\sin\theta = \sin\phi + \sin\psi,$$
$$\cos\theta = -(\cos\phi + \cos\psi).$$

Squaring and adding, we find

$$\cos(\psi - \phi) = -\tfrac{1}{2},\quad \text{or}\quad \psi - \phi = 120^\circ.$$

That is, the angle $BOC = 120^\circ$; and in the same way it may be shewn that $AOC = 120^\circ = AOB$. Hence if on any two sides of the triangle we describe segments of circles containing angles of 120°, their intersection will determine the point O. The actual length of the lines u, v, w, and the value of the minimum sum may be found. For from the geometry of the figure we have the equations

$$v^2 + vw + w^2 = a^2,$$
$$u^2 + uw + w^2 = b^2,$$
$$u^2 + uv + v^2 = c^2,$$
$$\text{and}\quad uv + uw + vw = \frac{4m^2}{3^{\frac{1}{2}}},$$

m^2 being the area of the triangle. Adding the sum of the first three equations to three times the last,

$$2(u + v + w)^2 = a^2 + b^2 + c^2 + 4\,.\,3^{\frac{1}{2}}m^2;$$

$$\text{whence}\quad u + v + w = \left\{\tfrac{1}{2}(a^2 + b^2 + c^2) + 2\,.\,3^{\frac{1}{2}}m^2\right\}^{\frac{1}{2}}.$$

Calling this r, and subtracting the first three equations two and two, we have

$$r(v - u) = a^2 - b^2,\quad r(w - v) = b^2 - c^2,\quad r(u - w) = c^2 - a^2;$$

$$\text{whence}\quad v = u + \frac{a^2 - b^2}{r},\quad w = u + \frac{a^2 - c^2}{r},$$

$$\text{and therefore}\quad u + v + w = r = 3u + \frac{2a^2 - b^2 - c^2}{r}.$$

From this we find

$$u = \frac{r}{3} + \frac{b^2 + c^2 - 2a^2}{3r},$$

$$v = \frac{r}{3} + \frac{a^2 + c^2 - 2b^2}{3r},$$

$$w = \frac{r}{3} + \frac{a^2 + b^2 - 2c^2}{3r}.$$

This problem possesses considerable interest in the history of mathematics. It was proposed by Fermat to Torricelli, who, after some time gave three solutions of it. He communicated it to Vincent Viviani, who also solved it, and gave the geometrical construction mentioned above; but he says that it is a problem "quod, ut vera fateor, non nisi iteratis oppugnationibus tunc nobis vincere datum fuit." For his demonstration see his *Geometrica Divinatio de Maximis et Minimis*, p. 150. The reader will also find a discussion of this problem, and of the more general one where the minimum function is $\alpha u + \beta v + \gamma w$, in a paper by Fuss in the *Nova Acta Petrop.* Vol. XI. p. 235.

CHAPTER VIII.

ON THE GENERATION OF CURVES AND THE INVESTIGATION OF THEIR EQUATIONS FROM THEIR GEOMETRICAL PROPERTIES.

As in the following chapters frequent reference will be made to certain curves which have acquired historical importance, and have in consequence been distinguished by particular names, I shall here describe the mode of their generation and deduce their equations from their definitions, adding some notice of the principal properties which possess interest.

(1) The Cissoid of Diocles.

This curve, named after Diocles, a Greek mathematician, who is supposed to have lived about the sixth century of our era, was invented by him for the purpose of constructing the solution of the problem of finding two mean proportionals. The curve is generated in the following manner: In the diameter ACB (fig. 12) of the circle $ADBE$ take $BM = AN$, erect the ordinates QM, RN and join AQ; the locus of the point P where the line AQ cuts the ordinate RN is the cissoid of Diocles. To find its equation, put $AN = x$, $PN = y$, $AC = a$: then as

$$\frac{PN}{AN} = \frac{QM}{AM}, \quad \frac{y}{x} = \frac{(2ax - x^2)^{\frac{1}{2}}}{2a - x},$$

$$\text{or } y^2 (2a - x) = x^3,$$

which is the equation to the curve.

The curve has an equal and similar branch on the other side of AB; the two branches meet in a cusp at the point A, and have the line HK as a common asymptote. The area included between the curve and the asymptote is three times the area of the generating circle.

The application of this curve to the solution of the problem of two mean proportionals is very simple. Pappus has shown that if *BC*, *CS* be the two quantities between which the two mean proportionals are to be inserted, and if the line *APQ* be drawn so that $QT = PT$, the line *CT* is the first of the two mean proportionals: it is obvious from this that *P* is a point in the cissoid. If therefore we wish to find two mean proportionals between *BC* and *CS*, we construct the cissoid *AQD* and produce *BS* till it meet the curve in a point *P*. Joining *AP*, and producing it to meet *CS* produced, we determine the line *CT* which is the first of the two mean proportionals required.

According to the geometrical ideas of the ancients a problem was not thought to be completely solved unless a mechanical construction was given. To complete therefore the theory of the cissoid, Newton* invented the following means of describing it by continuous motion. At the centre *C* of the circle *ADB* (fig. 13) erect the perpendicular *CDE*, of indefinite length. Take a point *O* in *CA* produced such that $AO = AC$; then if the rectangular ruler *NLM*, of which the leg *LM* is equal to the diameter of the circle, be moved so that the leg *NL* always slides along *O*, while the end *M* slides along *CDE*, the middle point *P* of *LM* will trace out the cissoid.

(2) The Conchoid of Nicomedes.

This curve, the invention of Nicomedes, who lived about the second century of our era, was, like the preceding, first formed for the purpose of constructing the solution of the problem of finding two mean proportionals, or the duplication of the cube, but it is more readily applicable to another problem not less celebrated among the ancients, that of the trisection of an angle. The curve is generated in the following manner: take the indefinite straight line *HK*, (fig. 14) and from a fixed point *O* draw a line *OMP* cutting the line *HK* in *M*; take the point *P*, such that *PM* shall be always of a constant length: the locus of the point *P* is the conchoid. The point *P* may be taken

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[*] Append. ad Arith. Univ.

between O and M, in which case it will trace out another branch of the curve which is called the inferior conchoid. To determine the equation, let $AN=x$, $PN=y$, PM (which is of constant length) $=a$, $OA=b$.

Then as $PM^2 = PN^2 + MN^2$, and

$$MN = AN - AM = AN - \frac{OA}{PN} MN,$$

we have $$x^2y^2 = (a^2 - y^2)(b+y)^2,$$

which is the equation to the curve, including both the superior and the inferior conchoid.

It is evident from the construction of the curve that the line KH is an asymptote to both branches. When $a>b$ there is a loop in the inferior conchoid at O as in the figure; when $a=b$ the loop degenerates into a cusp; and when $a<b$ there are two points of contrary flexure, one on each side of the line OA.

The application of this curve to the construction of the problem of the trisection of an angle is as follows. It may be readily shewn, that if AOB (fig. 15) be the angle to be trisected, and if the line OMP be so drawn that the part MP, intercepted between AB and BC at right angles to each other, is double of OB, the angle AOM is the third part of AOB. Now if we describe a conchoid with O as pole and the line AB as directrix, the constant parameter being equal to twice OB, its intersection with BC will determine the point P.

Nicomedes appears to have been led to the invention of this curve as a means of solving the celebrated problems mentioned above, by the facility with which it could be constructed mechanically. For if we take a grooved rule HK (fig. 16) and another grooved rule PQ, having a fixed pin at a point M, and bearing a pencil at P, and if we cause the pin at M to slide along the groove HK while the groove MQ slides along a pin fixed at O, the point P will trace out the conchoid.

(3) The Witch of Agnesi.

In the ordinate produced of the circle AMB (fig. 17) take a point P, such that $PN : AB = MN : AN$; the locus of the point P is the curve called the Witch.

Putting $AC = a$, $AN = x$, $PN = y$, we find as the equation to the curve

$$xy^2 = 4a^2 (2a - x).$$

This curve is given by Donna Maria Agnesi in her *Instituzioni Analitiche*, Art. 238, and is called by her the "Versiera."

The line KAH is an asymptote to the curve, which has two points of contrary flexure corresponding to $x = \frac{3a}{2}$.

(4) The Lemniscate of Bernoulli.

If a point be taken such that the product of the lines drawn from it to two fixed points is constant, it will trace out the curve called the lemniscate*. If $2a$ be the distance between the fixed points, and if the origin be taken at the middle point between them, the equation to the curve is

$$\{y^2 + (a + x)^2\} \{y^2 + (a - x)^2\} = c^4.$$

When $c = a$, the equation is reduced to

$$(x^2 + y^2)^2 = 2a^2 (x^2 - y^2).$$

This was the curve used by James Bernoulli† in the construction of the curve along which a body under the action of gravity will advance or recede uniformly from a fixed point.

It is the locus of the intersections of tangents to a rectangular hyperbola with perpendiculars drawn to them from the centre, and its form is that of the figure ∞. Of the properties of the arcs of this curve, which have been investigated by Fagnani and Euler, we shall treat in the chapter on the comparison of Transcendants in the Integral Calculus.

If we assume $x = r\cos\theta$, $y = r\sin\theta$, we find

$r^2 = 2a^2 \cos 2\theta$ as the polar equation to the curve.

(5) The Logarithmic Curve.

The definition of this curve is that the abscissa is proportional to the logarithm of the ordinate. Hence its equation is $y = b\epsilon^{\frac{x}{c}}$,

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* From *lemniscus*, a ribbon.

† *Opera*, p. 609.

or, as it is generally written, $y = a^x$.

The subtangent is constant, and the axis of x is an asymptote. The whole area included between the curve, the axis of x and any ordinate is equal to twice the triangle formed by the ordinate, the tangent at its extremity and the axis of x; and the solid formed by the revolution of the curve round its asymptote is equal to a cylinder, the radius of whose base is the bounding ordinate, and whose height is the tangent at its extremity.

This curve was invented by James Gregorie*, who investigated some of its properties: others were discovered by Huyghens. Euler†, and more recently Vincent, in the *Annales de Gergonne*, Vol. xv. p. 1, have conceived that the equation $y = a^x$ expresses, besides the continuous curve, a series of discontinuous points, forming what the latter calls a "courbe pointillée." This conclusion appears to me to be founded on an erroneous conception of the principles of the interpretation of algebraical expressions, and I have elsewhere‡ stated my reasons for believing that these discontinuous points belong each to a separate continuous curve which does not lie in the plane of reference, and that they cannot be properly included in the equation to one curve. As however the question is more closely connected with the analytical Theory of Logarithms than with the subject of which we here treat, I shall not now enter into the argument, but shall content myself with referring the reader, who is curious in such matters, to the papers quoted above, and to De Morgan's *Differential Calculus*, p. 383, where he will find the views of Vincent supported and illustrated§.

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* *Geometriæ Pars Universalis, Pref.*

† *Introductio in Analysin Infinitorum*, Vol. II. p. 290.

‡ *Camb. Math. Journal*, Vol. I. p. 231, and p. 264.

§ Professor De Morgan says, "that those who object to the pointed branch as introducing discontinuity, must choose between its discontinuity and that of an abrupt termination." It appears to me that if we interpret our analytical symbols with proper generality so as to introduce those branches of curves which do not lie in the plane of reference, we avoid the second horn of his dilemma.

(6) The Catenary.

This is the curve which a perfectly flexible chain will assume when suspended from two points in the same horizontal line; I must refer the reader to works on statics for an investigation of its equation, which is

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right).$$

Its most important geometrical properties are analogous to properties of the circle. Thus, the part of the normal intercepted between the curve and the axis of x is equal to the radius vector, but measured in the opposite direction; and if we represent an ordinate corresponding to the abscissa x by $f(x)$, and the corresponding area by $cF(x)$, we shall readily find from the preceding equation that

$$cf(x+x') = f(x)f(x') + F(x)F(x'),$$

$$cf(x-x') = f(x)f(x') - F(x)F(x'),$$

$$cF(x+x') = F(x)f(x') + f(x)F(x'),$$

$$cF(x-x') = F(x)f(x') - f(x)F(x').$$

It is obvious that the preceding formulæ are analogous to those connecting sines and cosines of circular arcs. For these and other properties of the catenary connected with the involute of the parabola, see a paper by Professor Wallace in the *Edinburgh Transactions*, Vol. XIV. p. 625.

(7) The Quadratrix of Dinostratus.

If the radius CQ of the circle ABD (fig. 18) revolve uniformly round C from A to B, while the ordinate NM also moves uniformly parallel to itself from A to C, the locus of their intersection will be the quadratrix of Dinostratus. To find its equation, let $AM = x$, $PM = y$, $AC = a$. Then from the uniformity of the motion of CQ and MN, we have

$$ACQ : ACB = AM : AC;$$

$$\text{whence } ACQ = \frac{\pi}{2}\frac{x}{a}.$$

But $PM = CM \tan ACQ$, therefore the equation to the curve is

$$y = (a - x) \tan \left(\frac{\pi}{2} \cdot \frac{x}{a}\right).$$

This curve was used by Dinostratus (a mathematician of the school of Plato) for the purpose of dividing an angle into any number of parts, and also of squaring the circle, from which it derives its name. The following is the property which enables the curve to be so employed.

When $x = a$, we have (by Chap. VI. Ex. 29) $y = CE = \frac{2a}{\pi}$, so that CE is a third proportional to the quadrant and the radius, and thus if the point E could be determined by means of the straight line and circle, the circle could be squared.

Léotaud, in his treatise on this curve appended to his *Cyclomathia*, showed that it is not confined within the semicircle ABD, but that it has two infinite branches extending below the axis of x, and bounded by asymptotes parallel to the axis of y at distances $-a$ and $3a$ from the origin. In addition to this, the curve has an infinite number of infinite branches, which are bounded by asymptotes parallel to the axis of y at distances $5a$, $7a$, &c., $-3a$, $-5a$, &c. from the origin, and which cut the axis of x at distances $4a$, $6a$, &c., $-2a$, $-4a$, &c. from the origin. The farther these points are removed from the origin the more nearly is the curve perpendicular to the axis of x, the value of $\frac{dy}{dx}$ at the intersection being $\pm (2n - 1) \frac{\pi}{2}$, $2na$ being the abscissa of the point where the curve cuts the axis of x.

Rolling Curves.

(8) The Cycloid.

This curve is generated by a point P in the circumference of a circle bPc (fig. 19), which rolls along a line AA'. To find its equation put

$$O'b = a, \quad PO'b = \theta,$$
$$AM = x, \quad PM = y.$$

Then $AM = Ab - Mb$, or $x = a(\theta - \sin\theta)$,

$PM = O'b + O'd$, or $y = a(1 - \cos\theta)$.

These two equations taken simultaneously represent the curve, or, if we eliminate θ between them, we obtain as its equation

$$x = a \operatorname{vers}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}}.$$

If we take C, the highest point of the curve as our origin, and put $CN = x$, $PN = y$, and $cO'p = \phi$, we should find

$$x = a(1 - \cos\phi), \qquad y = a(\phi + \sin\phi);$$

whence $y = a \operatorname{vers}^{-1}\dfrac{x}{a} + (2ax - x^2)^{\frac{1}{2}}$.

It is easy to see both from geometrical and analytical considerations that the cycloid is not limited to the space between A and A', but that it consists of an infinite number of portions equal and similar to ACA' and touching each other in cusps as in the figure.

After the Conic Sections there is no curve in geometry which has more exercised the ingenuity of mathematicians than the cycloid, and their labours have been rewarded by the discovery of a multitude of interesting properties, important both in geometry and in dynamics.

The invention of this curve is usually ascribed to Galileo, but Wallis in a letter to Leibnitz* says, that it is mentioned by Cardinal de Cusa in a work published in 1510, and that in the MSS. the date of which is about 1454, it is "pulchrè delineatam", therein differing from the printed copies. Roberval proved that the whole area of the cycloid is three times that of the generating circle, and this discovery, which was the cause of many disputes between rival claimants to the honour of making it, drew the attention of mathematicians to the study of the properties of this new curve. Among others, Descartes occupied himself with the subject, and he

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* *Leibn. Opera*, Vol. III. p. 95.

showed how to draw tangents to the curve, and proved that the tangent at any point P (fig. 19) is perpendicular to the corresponding chord BQ of the generating circle, and consequently that it is parallel to CQ: from this also it readily follows that if QR be a tangent to the generating circle at Q, $QR = PQ =$ arc BQ. Wren was the first who rectified the cycloid, and he showed that the length of an arc measured from the vertex is equal to twice the chord of the generating circle which is parallel to the tangent at the extremity, so that the whole length of the curve is equal to four times the diameter of the generating circle. Pascal discovered the means of finding the area and the centre of gravity of any segment of the curve as well as the content and surface of the solids formed by the revolution of the segment round the axis of the curve, and the base of the segment, and to the solution of these problems he challenged all mathematicians in a letter which he circulated under the name of Dettonville, offering at the same time a prize of forty pistoles to the first and one of twenty pistoles to the second person who should solve them. Wallis and Lalouère appeared as candidates for the prize, but none was awarded. To Huyghens is due the discovery that the evolute of the cycloid is an equal cycloid in an inverted position, and that the radius of curvature is double of the chord of the generating circle which is perpendicular to the tangent. He also discovered the important dynamical property of the tautochronism of a cycloidal pendulum; that is to say, that a body under the action of gravity falling down an inverted cycloid with its base horizontal, will reach the lowest point in the same time from whatever point it begins to fall. Two of the most remarkable properties of this curve were discovered by John Bernoulli: 1st, that it is the curve along which a body will, under the action of gravity, fall in the shortest time from one given point to another not in the same vertical: 2nd, that if any arc of a curve as AB (fig. 21), the tangents at the extremities of which are at right angles to each other, be evolved into a curve BA', beginning from B: and if the same operation be performed on $A'B$, beginning from A', and so on in succession, the successive involutes will continually approximate to a common cycloid, the axis of which is parallel to AC*. The preceding are only a few of the most important properties of this curve; for a detailed account of all which the industry of mathematicians has discovered, the reader must be referred to the treatises on the cycloid which have been written by various authors. Such are the *Histoire de la Roulette* of Pascal; the *History of the Cycloid* of Carlo Dati; the Treatise *de Cycloide* of Wallis; the *Historia Cycloidis* of Groningius in his *Bibliotheca Universalis;* and the work of Lalouère called *Geometria promota in* VII *de Cycloide libris.*

(9) The Companion to the Cycloid.

If the ordinate QN (fig. 20) of a semicircle be produced till it be equal to the arc CQ, its extremity will lie in a curve which is called the companion to the cycloid. The co-ordinates of a point in this curve are, putting $CO = a$, $CN = x$, $CN = y$, $COQ = \theta$,

$$x = a(1 - \cos\theta), \quad y = a\theta.$$

It has points of contrary flexure at the extremities D and d of an ordinate passing through the centre of the generating circle. The space COD is equal to the square of the radius; the whole area ACa is equal to twice that of the generating circle, and if the line AC be drawn, the area AMD is equal to the area CLD.

(10) If instead of supposing the point P to be in the circumference of the generating circle we suppose it to be either within the area of the circle or without it, the curve traced out is called a Trochoid. The equations to such a curve are

$$x = a(\theta - n\sin\theta),$$

$$y = a(1 - n\cos\theta),$$

where n is the ratio of the distance of the tracing point from the centre of the generating circle to the radius of that circle.

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* John Bernoulli, *Opera*, Vol. IV. p. 98. Euler, *Commen. Petrop.* 1766. Legendre, *Exercices du Calcul Integral*, Tom. II. p. 491.

(11) Epitrochoids and Hypotrochoids.

When the generating circle rolls, not on a straight line, but on the circumference of another circle, the curve generated is called an Epitrochoid or a Hypotrochoid, according as the curve rolls on the exterior or interior of the fixed circle. Let O (fig. 22) be the centre of the fixed circle, C that of the generating circle, a, b their radii. Let A and Q be the points originally in contact, P the tracing point. Then if we make $CP = h$, $CN = x$, $PN = y$, $AOB = \theta$, so that $QCB = \frac{a}{b}\theta$, we find

$$x = OH + HN = (a+b)\cos\theta - h\cos\left(\frac{a+b}{b}\right)\theta,$$

$$y = CH - CK = (a+b)\sin\theta - h\sin\left(\frac{a+b}{b}\right)\theta.$$

If we suppose the generating circle to roll in the inside of the fixed circle as in fig. 23, we should find

$$x = (a-b)\cos\theta + h\cos\left(\frac{a-b}{b}\right)\theta,$$

$$y = (a-b)\sin\theta - h\sin\left(\frac{a-b}{b}\right)\theta.$$

When $h = b$ these become the equations to the Epicycloid and Hypocycloid respectively. When a and b are commensurable the curve will re-enter after a number of revolutions of the generating circle equal to the least common multiple of a and b: in such cases the curve is expressible by an algebraical equation between x and y. When a and b are incommensurable the curve will never re-enter, and is expressible only by some transcendental equation between x and y.

If $h = b$ and $b = a$ the equations to the epicycloid are

$$x = a(2\cos\theta - \cos 2\theta),$$

$$y = a(2\sin\theta - \sin 2\theta),$$

or

$$x = a\{1 + 2\cos\theta(1 - \cos\theta)\},$$

$$y = 2a\sin\theta(1 - \cos\theta).$$

Whence, squaring and adding,

$$x^2 + y^2 = a^2 \left\{1 + 4(1 - \cos\theta)\right\}.$$

But we have also

$$(x - a)^2 + y^2 = 4a^2 (1 - \cos\theta)^2.$$

Therefore $(x^2 + y^2 - a^2)^2 = 4a^2 \left\{(x - a)^2 + y^2\right\}$ is the equation to the curve expressed in rectangular coordinates. If we put $x = a + r\cos\phi$, $y = r\sin\phi$,

we find $r = 2a(1 - \cos\phi)$,

as the polar equation. From its shape this curve is called the Cardioid: in common with the circle it possesses the property that all lines drawn through its pole and bounded both ways by the curve are of equal length.

In the equations to the hypotrochoid, if we make $h = b$ and $b = \frac{a}{4}$, we obtain by the elimination of θ the equation

$$(a^2 - x^2 - y^2)^3 = 27\, a^2 x^2 y^2;$$

which may be put under the form $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

This hypocycloid occurs in the solution of many problems.

If in the equations to the hypotrochoid we put $b = \frac{a}{2}$, then

$$x = \left(\frac{a}{2} + h\right)\cos\theta, \quad y = \left(\frac{a}{2} - h\right)\sin\theta.$$

Whence $\left(\frac{a}{2} - h\right)^2 x^2 + \left(\frac{a}{2} + h\right)^2 y^2 = \left(\frac{a^2}{4} - h^2\right)^2$,

which is the equation to an ellipse the axes of which are

$$\frac{a}{2} + h \quad \text{and} \quad \frac{a}{2} - h.$$

If $h = \frac{a}{2}$ the hypocycloid becomes a straight line, which is one of the diameters of the fixed circle.

Professor Wallace* has made a very elegant application of the preceding property of the ellipse to generate that curve by continuous motion. *A* and *B* (fig. 24) are two wheels the axes of which turn in holes *O*, *C* near the ends of the connecting bar *OC*. The diameter of the wheel *B* is one half of that of *A*, and a band *EF* goes round them. An arm *CP* is attached to the wheel *B*, and bears at its extremity *P* a tracing pencil. If now the wheel *A* be fixed while the bar *OC* is turned round *O*, the wheel *B* will, by the action of the band, be made to revolve twice round its centre, while the bar revolves once round *O*: the point *P* will then trace out an ellipse.

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* Wallace's *Conic Sections*, p. 182.

All *Epicycloids* and *Hypocycloids* are rectifiable, as was first shown by Newton*. The length of the arc of the epicycloid comprised between two contiguous cusps—that is, the length of the arc produced by one revolution of the generating circle—is $\frac{4b}{a}(a+b)$, and the corresponding arc of the hypocycloid is $\frac{4b}{a}(a-b)$.

The corresponding area of the epicycloid is $\frac{\pi b^2}{a}(3a+2b)$ and of the hypocycloid it is $\frac{\pi b^2}{a}(3a-2b)$.

The evolute of the epicycloid is a similar figure, the radii of the fixed and generating circles being $\frac{a^2}{a+2b}$ and $\frac{ab}{a+2b}$ respectively. An analogous theorem holds for the hypocycloid.

(12) The Spiral of Archimedes.

While the straight line *OM* (fig. 25) revolves uniformly round *O*, let the point *P* move uniformly along *OM*: the locus of the point *P* is the spiral of Archimedes. To find its equation let $AOP = \theta$, $OP = r$, and when $\theta = 2\pi$ let $r = a$.

Then $$\frac{r}{\theta} = \frac{a}{2\pi}, \quad \text{or } r = \frac{a}{2\pi}\theta,$$

which is the equation to the curve.

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[*] *Principia* I. Prop. 49.

The following are its principal properties. The area of any sector bounded by a line as $OQ = r$ is one third of the circular sector QOR, and it is one half of the area of the segment of a parabola $\left(\text{whose latus rectum is } \frac{a}{\pi}\right)$ included between the vertex and an ordinate $= r$. The length of the arc of the sector of the spiral is equal to that of the segment of the parabola. If a tangent be drawn at the extremity of the arc formed by one revolution of the radius, the subtangent will be equal to the circumference of the circle whose radius is a. If at the extremity of the arc formed by two revolutions, it will be double of the circumference, and so on.

This curve was invented by Conon, but its principal properties were discovered by the geometer whose name it bears.

(13) The Logarithmic Spiral.

The definition of this spiral is, that the radius increases in a geometric while the angle increases in an arithmetic ratio. Hence its equation will be of the form $r = c\epsilon^{\frac{\theta}{a}}$,

or, as it is usually written, $r = a^{\theta}$.

This curve was imagined by Descartes, who also noticed two of its properties; that at every point it makes equal angles with the tangent, and that the length of the curve measured from the origin is proportional to the radius of its extremity. Since $r = 0$ when $\theta = -\infty$, it appears that the curve makes an infinite number of revolutions before it reaches the pole; a property which was at first disputed by Descartes. From the form of the equation it is easy to see that radii including equal angles are proportional: for if

$$r = a^{\theta} \text{ and } r_1 = a^{\theta+\alpha}, \quad \frac{r_1}{r} = a^{\alpha}.$$

Again, if $\rho = a^{\phi}$, $\rho_1 = a^{\phi+\alpha}$, $\frac{\rho_1}{\rho} = a^{\alpha}$;

and therefore $\frac{r_1}{r} = \frac{\rho_1}{\rho}$.

The length of an arc of the curve measured from the pole is equal to the portion of the tangent at its extremity cut off by the subtangent, and the area is one half of the triangle contained by the bounding radius, the tangent at its extremity, and the subtangent. But the most remarkable properties of this curve were discovered by James Bernoulli, who showed* that this spiral can be made to reproduce itself in many ways. The evolute and involute of this curve are both spirals equal to the original one, and differing from it in position only; its caustics both by reflexion and refraction (the pole being the origin of light) are also spirals equal to the primary one; and if another equal spiral be made to roll on the first, the pole of the rolling spiral will trace out another spiral equal to the original. This property of the logarithmic spiral of constantly reproducing itself appeared so remarkable to Bernoulli that he called it *spira mirabilis*, and he was pleased to see in it a type of constancy amidst changes and in adversity, and a symbol of the resurrection. As a specimen of the fanciful light in which he viewed the properties of this curve, I add the concluding paragraph of his paper. "Cum autem ob proprietatem tam singularem tamque admirabilem mire mihi placeat spira hæc mirabilis, sic ut ejus contemplatione satiari vix queam; cogitavi illam ad res varias symbolice repræsentandas non inconcinne adhiberi posse. Quoniam enim semper sibi similem et eandem spiram gignit, utcunque volvatur, evolvatur, radiet; hinc poterit esse vel sobolis parentibus per omnia similis Emblema: *Simillima filia matri.* ... Aut, si mavis, quia curva nostra mirabilis in ipsa mutatione semper sibi constantissime manet similis et numero eadem, poterit esse vel fortitudinis et constantiæ in adversitatibus; vel etiam carnis nostræ, post varias alterationes et tandem ipsam quoque mortem, ejusdem numero resurrecturæ symbolum; adeo quidem ut si Archimedem imitandi hodienum consuetudo obtineret libenter spiram hanc tumulo meo juberem incidi cum epigraphe: *Eadem mutata resurget.*"

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* *Opera*, p. 497.

CHAPTER IX.

ON THE TANGENTS, NORMALS AND ASYMPTOTES TO CURVES.

SECT. 1. *Rectilinear Co-ordinates.*

If the equation to the curve be put under the form

$$y = f(x),$$

the equation to a tangent at a point xy is

$$y' - y = \frac{dy}{dx}(x' - x);$$

x' and y' being the current co-ordinates of the tangent.

If the equation to the curve be put under the form

$$u = \phi(x, y) = c,$$

the equation to the tangent takes the more symmetrical form

$$\frac{du}{dx}(x' - x) + \frac{du}{dy}(y' - y) = 0.$$

If u be a homogeneous function of n dimensions in x and y, by a well-known property of such functions

$$x\frac{du}{dx} + y\frac{du}{dy} = nu = nc,$$

and the equation to the tangent becomes

$$x'\frac{du}{dx} + y'\frac{du}{dy} = nc.$$

The equations to the normal are

$$y' - y = -\frac{dx}{dy}(x' - x);$$

$$\text{or } \frac{du}{dx}(y' - y) - \frac{du}{dy}(x' - x) = 0.$$

The length of the subtangent is $y\dfrac{dx}{dy}$.

The length of the subnormal is $y\dfrac{dy}{dx}$.

The length of the tangent is $y\left\{1+\left(\dfrac{dx}{dy}\right)^2\right\}^{\frac{1}{2}}$.

The length of the normal is $y\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$.

The perpendicular from the origin on the tangent is

$$p=\frac{ydx-xdy}{(dx^2+dy^2)^{\frac{1}{2}}}=\frac{x\dfrac{du}{dx}+y\dfrac{du}{dy}}{\left\{\left(\dfrac{du}{dx}\right)^2+\left(\dfrac{du}{dy}\right)^2\right\}^{\frac{1}{2}}}=\frac{nc}{\left\{\left(\dfrac{du}{dx}\right)^2+\left(\dfrac{du}{dy}\right)^2\right\}^{\frac{1}{2}}},$$

if u be a homogeneous function of n dimensions in x and y.

The portion of the tangent intercepted between the point of contact and the perpendicular on it from the origin is

$$t=\frac{xdx+ydy}{(dx^2+dy^2)^{\frac{1}{2}}}=\frac{x\dfrac{du}{dy}-y\dfrac{du}{dx}}{\left\{\left(\dfrac{du}{dx}\right)^2+\left(\dfrac{du}{dy}\right)^2\right\}^{\frac{1}{2}}}.$$

The portions of the axes cut off between the origin and the tangent, or the intercepts of the tangent, are

$$y-x\frac{dy}{dx} \text{ along the axis of } y,$$

$$x-y\frac{dx}{dy} \text{ along the axis of } x.$$

These I shall call y_0, x_0 respectively.

Ex. (1). The equation to the hyperbola referred to its asymptotes is

$$xy=m^2.$$

Then $\dfrac{du}{dx}=y$, $\dfrac{du}{dy}=x$, and the equation to the tangent is

$$y(x'-x)+x(y'-y)=0;$$

$$\text{or}\quad yx'+xy'=2xy=2m^2.$$

Since $\dfrac{dy}{dx} = -\dfrac{m^2}{x^2}$, the subtangent $= y\dfrac{dx}{dy} = -\dfrac{x^2 y}{m^2} = -x$, as $xy = m^2$.

The perpendicular on the tangent $p = \dfrac{2m^2}{(x^2+y^2)^{\frac{1}{2}}}$.

Also $y_0 = y + \dfrac{m^2}{x} = 2y$, and $x_0 = x + \dfrac{m^2}{y} = 2x$.

Hence the product of the intercepts of the tangent

$$= x_0 y_0 = 4xy = 4m^2 \text{ is constant;}$$

and the triangle contained between the axes and the tangent, being proportional to this product, is also constant.

(2) The equation to the parabola referred to two tangents as axes is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1.$$

Hence the equation to the tangent is

$$\frac{x'}{(ax)^{\frac{1}{2}}} + \frac{y'}{(by)^{\frac{1}{2}}} = 1.$$

The intercepts are $x_0 = (ax)^{\frac{1}{2}}$, $y_0 = (by)^{\frac{1}{2}}$;

$$\text{therefore } \frac{x_0}{a} + \frac{y_0}{b} = \left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1;$$

or x_0, y_0 are the co-ordinates of the chord joining the points at which the axes touch the curve.

(3) The equation to one of the hypocycloids referred to rectangular co-ordinates is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

The equation to the tangent is

$$\frac{x'}{x^{\frac{1}{3}}} + \frac{y'}{y^{\frac{1}{3}}} = a^{\frac{2}{3}}.$$

Therefore $x_0 = a^{\frac{2}{3}}x^{\frac{1}{3}}$, $y_0 = a^{\frac{2}{3}}y^{\frac{1}{3}}$; and the portion of the tangent intercepted between the axes $= (x_0^2 + y_0^2)^{\frac{1}{2}} = a$; or the hypocycloid is constantly touched by a straight line of given length which slides between two rectangular axes. The converse of this proposition, viz. that the locus of the ultimate intersections of a line of given length sliding between rectangular axes is this hypocycloid, was first shown by John Bernoulli. (See his Works, Vol. III. p. 447.)

For the perpendicular from the origin on the tangent we find

$$p = (axy)^{\frac{1}{3}}.$$

(4) In the cissoid of Diocles,

$$y^2 = \frac{x^3}{2a - x};$$

$$\text{whence the subtangent } = \frac{x(2a - x)}{3a - x};$$

$$\text{and the subnormal } = \frac{x^2(3a - x)}{(2a - x)^2}.$$

(5) In the logarithmic curve

$$y = ce^{\frac{x}{a}}.$$

The subtangent $= a$, and is therefore constant.

The tangent $= (a^2 + y^2)^{\frac{1}{2}}$.

The subnormal $= \dfrac{y^2}{a}$. The normal $= \dfrac{y}{a}(a^2 + y^2)^{\frac{1}{2}}$.

$$p = \frac{y(a - x)}{(a^2 + y^2)^{\frac{1}{2}}}, \quad t = \frac{y^2 + ax}{(a^2 + y^2)^{\frac{1}{2}}}.$$

(6) In the catenary

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right).$$

The subnormal $= \dfrac{c}{4}\left(\epsilon^{\frac{2x}{c}} - \epsilon^{-\frac{2x}{c}}\right)$. The normal $= \dfrac{y^2}{c}$.

The subtangent $= \dfrac{cy}{(y^2 - c^2)^{\frac{1}{2}}}$. The tangent $= \dfrac{y^2}{(y^2 - c^2)^{\frac{1}{2}}}$.

(7) From the general parabolic equation

$$y^m = a^{m-1} x,$$

we find the equation to the tangent to be

$$mx(y' - y) = y(x' - x).$$

The subtangent $= mx$. The subnormal $= \dfrac{y^2}{mx} = \dfrac{a^{m-1}}{my^{m-2}}$,

$$y_0 = \frac{m-1}{m} y, \qquad x_0 = -(m-1)x,$$

whence $m^m y_0^m = -(m-1)^{m-1} a^{m-1} x_0$,

$$p = \frac{(m-1)xy}{(m^2x^2 + y^2)^{\frac{1}{2}}}, \qquad t = \frac{mx^2 + y^2}{(m^2x^2 + y^2)^{\frac{1}{2}}}.$$

(8) In the curve $x = \epsilon^{\frac{x-y}{y}}$ we easily find, by taking the logarithmic differential,

$$y_0 = \frac{y^2}{x}, \qquad x_0 = \frac{xy}{y - x}.$$

$$\text{Subtangent} = \frac{x^2}{x - y} = -y\frac{x_0}{y_0} = -x_0\frac{x}{y}.$$

(9) The equation to the cycloid referred to its vertex is

$$\frac{dy}{dx} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}},$$

AB (fig. 19) being the axis of x.

If M be the point where the ordinate meets the generating circle, and if we join MA, MB, then

$$\tan MAN = \frac{MN}{AN} = \frac{(2ax - x^2)^{\frac{1}{2}}}{x} = \frac{dy}{dx}.$$

That is to say, the tangent to the cycloid is parallel to the chord of the generating circle. The normal is evidently parallel to the other chord MB. Hence also the angle which two tangents make with each other is equal to the angle between the corresponding chords of the generating circle.

$$y_0 = y - (2ax - x^2)^{\frac{1}{2}} = PN - MN = PM.$$

But from the generation of the curve, PM is equal to the arc of the circle AM, therefore $y_0 = \text{arc } AM$.

Also $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \left(\dfrac{2a}{x}\right)^{\frac{1}{2}} = \dfrac{(2ax)^{\frac{1}{2}}}{x}$,

and chord $AM = (2ax)^{\frac{1}{2}}$. Therefore

$$\text{normal} = \frac{y}{x}(2ax)^{\frac{1}{2}} = \frac{PN.AM}{AN},$$

$$p = \frac{xy_0}{(2ax)^{\frac{1}{2}}} = AN.\frac{\text{arc } AM}{\text{chord } AM}.$$

(10) If p, r, be the perpendicular on the tangent and the radius vector at any point of a curve, then $\dfrac{p^2}{r}$ will be the perpendicular on the tangent at the corresponding point of the curve which is the locus of the extremity of p.

Let x, y, be the co-ordinates of the first curve, α, β, of the second; then p being the perpendicular on the tangent, its equation is

$$\alpha x + \beta y = p^2 = \alpha^2 + \beta^2, \tag{1}$$

since α, β, are the co-ordinates of the extremity of p. But the line being a tangent, this equation will hold when we put $x + dx$ and $y + dy$ for x and y; we then have

$$\alpha\, dx + \beta\, dy = 0. \tag{2}$$

Now if $V = 0$ be the equation connecting α and β, that is to say, the equation to the locus of the extremity of p, and if P be the perpendicular on the tangent of that curve,

$$P = \frac{\alpha \dfrac{dV}{d\alpha} + \beta \dfrac{dV}{d\beta}}{\left\{\left(\dfrac{dV}{d\alpha}\right)^2 + \left(\dfrac{dV}{d\beta}\right)^2\right\}^{\frac{1}{2}}}.$$

But from the equation to the curve

$$\frac{dV}{d\alpha}\, d\alpha + \frac{dV}{d\beta}\, d\beta = 0. \tag{3}$$

Now differentiating (1) considering x, y, α, β, as variables, and paying attention to (2), we have

$$(x - 2\alpha)\, d\alpha + (y - 2\beta)\, d\beta = 0. \tag{4}$$

$\lambda\,(3) - (4) = 0$ gives, on equating to zero the coefficients of each differential,

$$\lambda \frac{dV}{d\alpha} = x - 2\alpha, \quad \lambda \frac{dV}{d\beta} = y - 2\beta.$$

Substituting these values of $\frac{dV}{d\alpha}$ and $\frac{dV}{d\beta}$ in the expression for P, it becomes

$$P = \frac{2(\alpha^2 + \beta^2) - (\alpha x + \beta y)}{[x^2 + y^2 + 4\{\alpha^2 + \beta^2 - (\alpha x + \beta y)\}]^{\frac{1}{2}}},$$

which by (1) is reduced to

$$P = \frac{\alpha^2 + \beta^2}{(x^2 + y^2)^{\frac{1}{2}}} = \frac{p^2}{r}.$$

(11) To find the least polygon of a given number of sides which will circumscribe a given oval figure.

Let AB, BC, CD, (fig. 26) be consecutive sides of the polygon. Produce AB, DC to meet in E, which take as origin, the axes being EA, ED. Then the position of BC must be such as to make BEC a maximum.

Now calling as before the intercepts of the tangent x_0, y_0,

$$x_0 = x - y\frac{dx}{dy}, \quad y_0 = y - x\frac{dy}{dx};$$

x and y being the co-ordinates of the point of contact P.

The area $BEC = \frac{1}{2}x_0 y_0 \sin E$, therefore

$$\left(x - y\frac{dx}{dy}\right)\left(y - x\frac{dy}{dx}\right) = \left(y - x\frac{dy}{dx}\right)^2 \frac{dx}{dy}$$

is to be a maximum, (neglecting the negative sign).

Differentiate with respect to x,

$$\left(y - x\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\frac{dx}{dy}\left(2x + y\frac{dx}{dy} - x\right) = 0.$$

The last factor alone gives a solution. From it we have

$$x = \frac{1}{2}\left(x - y\frac{dx}{dy}\right) = \frac{1}{2}x_0.$$

That is, $EM = \frac{1}{2} EB = MB$, and hence also $CP = PB$, or CB is bisected at the point of contact. As the same condition holds for every side of the polygon, it follows that, when the polygon circumscribing an oval is a minimum, each side is bisected at the point of contact. Hence we see that of all the parallelograms which circumscribe an ellipse, those are least which have their sides parallel to conjugate diameters.

(12) The degree of a curve being n, there cannot be more than $n(n-1)$ tangents drawn to it from one point.

Let

$$u = c, \tag{1}$$

be the equation to the curve, then the equation to the tangent is

$$x' \frac{du}{dx} + y' \frac{du}{dy} = x \frac{du}{dx} + y \frac{du}{dy};$$

and the condition that this tangent shall pass through a given point a, b, is

$$a \frac{du}{dx} + b \frac{du}{dy} = x \frac{du}{dx} + y \frac{du}{dy}. \tag{2}$$

The equations (1) and (2) being combined together will give the values of x and y at the points of contact; and as both equations are of n dimensions in x and y, (since u is of n dimensions and $\frac{du}{dx}$ and $\frac{du}{dy}$ of $n-1$, and therefore $x \frac{du}{dx} + y \frac{du}{dy}$ of n dimensions), it would appear that the resulting equation is of the degree n^2, and therefore that there are as many tangents passing through the point. But the degree of the equation can always be reduced; for we may combine (2) with any multiple of (1), and the result of the elimination between the new equation and either of the others will still give us the co-ordinates of the point of contact. Multiply (1) by n and subtract it from (2), then we have

$$a \frac{du}{dx} + b \frac{du}{dy} - nc = x \frac{du}{dx} + y \frac{du}{dy} - nu. \tag{3}$$

Now by a property of homogeneous functions, if v be homogeneous of n dimensions in x and y,

$$x \frac{dv}{dx} + y \frac{dv}{dy} = nv.$$

This then will be true of the terms of n dimensions in u, and they will therefore disappear from the second side of the equation (3), which will thus be reduced to $(n-1)$ dimensions, since $\frac{du}{dx}$ and $\frac{du}{dy}$ are only of that degree. Hence the combination of (1) with (3) will rise only to the degree $n(n-1)$, which therefore represents the greatest number of tangents which can be drawn from a given point to a curve of n dimensions. Waring had fixed the limit at n^2, as it at first sight appears to be; the preceding process of reduction is due to Bobillier, *Annales de Gergonne*, Vol. XIX. p. 106. It is to be observed that though $n(n-1)$ is the greatest number of tangents which can be drawn, it seldom reaches that limit, since the final equation generally involves impossible roots which refer to tangents drawn to the branches of the curve which do not lie in the plane xy. Since $n(n-1)$ is essentially even, it may happen that for certain positions of the point all the roots are impossible; a result which is geometrically apparent, inasmuch as from the interior of an oval curve, such as the ellipse, no tangents can be drawn to the part of the curve which lies in the plane of xy.

Asymptotes.

As an asymptote is a line which, intersecting the axes at a finite distance from the origin, is a tangent to the curve at an infinite distance, it appears that if x_0 or y_0 remain finite when x or y are infinite, their values will determine the position of the asymptote.

A more convenient method however is that first given by Stirling, in his *Lineæ Tertii ordinis Newtonianæ,* p. 48.

If $y = f(x)$ be the equation to the curve, and if we can expand $f(x)$ in descending powers of x, so that

$$y = a_m x^m + a_{m-1} x^{m-1} + \&c. + a_1 x + a_0 + \frac{a_{-1}}{x} + \frac{a_{-2}}{x^2} + \&c.;$$

then when $x = \infty$, the terms involving negative powers of x vanish, and the equation to the curve coincides with that to another curve the equation to which is

$$y = a_m x^m + a_{m-1} x^{m-1} + \&c. + a_1 x + a_0.$$

This then is the general equation to a curvilinear asymptote, the nature of which will depend on the degree of the highest power of x which is involved in it. The most important case is that in which the equation is reduced to

$$y = a_1 x + a_0,$$

that is, in which the asymptote is a straight line.

This method fails when the asymptote is parallel to the axis of y, as in that case the coefficient of x would be infinite: but asymptotes of this kind are visible by a simple inspection of the equation to the curve when it is put under the form $y = f(x)$. For the value of y being infinite for the abscissa corresponding to the asymptote, we have only to find what values of x will make $f(x) = \infty$, or to make the denominator of $f(x)$ vanish, since no finite value of x in the numerator can make $f(x) = \infty$. These values of x being found, the ordinates drawn through them are asymptotes to the curve.

(13) Let the equation to the curve be

$$y^3 = ax^2 + x^3.$$

Then $$\frac{dy}{dx} = \frac{2ax + 3x^2}{3y^2},$$

and $$y_0 = y - \frac{2ax^2 + 3x^3}{3y^2} = \frac{3(y^3 - x^3) - 2ax^2}{3y^2}.$$

But from the equation to the curve, $3(y^3 - x^3) = 3ax^2$,

therefore $$y_0 = \frac{a}{3} \frac{x^2}{y^2}.$$

To find the value of $\frac{x^2}{y^2}$ when x and y are infinite, we have from the original equation

$$\frac{y^3}{x^3} = \frac{a}{x} + 1 = 1 \text{ when } x \text{ and } y \text{ are infinite.}$$

Therefore also $\frac{x^2}{y^2} = 1$ when x and y are infinite,

and hence $y_0 = \frac{a}{3}$.

Similarly, $$x_0 = x - \frac{3y^3}{2ax + 3x^2} = -\frac{ax^2}{2ax + 3x^2}$$

$$= -\frac{a}{3} \text{ when } x = \infty.$$

Hence the asymptote cuts the axis of y at a distance $\frac{a}{3}$, and that of x at a distance $-\frac{a}{3}$ from the origin, and as it is therefore inclined at an angle of 45° to the axis of x, its equation is

$$y = x + \frac{a}{3}.$$

(14) Let the equation to the curve be

$$y^2 = \frac{x^3 + ax^2}{x - a}.$$

Then $$y^2 = x^2\left(\frac{x + a}{x - a}\right) = x^2\left(\frac{1 + \frac{a}{x}}{1 - \frac{a}{x}}\right)$$

$$= x^2\left(1 + \frac{2a}{x} + \frac{2a^2}{x^2} + \&c.\right);$$

and $$y = \pm x\left(1 + \frac{a}{x} + \frac{a^2}{x^2} + \&c.\right).$$

Therefore $y = \pm(x + a)$ are the equations to two asymptotes at right angles to each other.

Another asymptote parallel to the axis of y is given by putting $x = a$.

(15) Let the equation to the curve be

$$y = \frac{x^3 - 3ax^2 + a^3}{x^2 - 3bx + 2b^2}.$$

The denominator equated to 0 gives $x = b$, $x = 2b$; therefore the corresponding ordinates are asymptotes, since for $x = b$ and $x = 2b$ y is infinite.

Also
$$y = \frac{x^3\left(1 - \frac{3a}{x} + \frac{a^3}{x^3}\right)}{x^2\left(1 - \frac{b}{x}\right)\left(1 - \frac{2b}{x}\right)}.$$

Whence, expanding and rejecting the terms involving negative powers of x, we have $y = x - 3(a - b)$ as the equation to a third asymptote, which is therefore inclined at an angle of 45° to the axis of x.

When the equation cannot be solved with respect to y, we are sometimes able to determine the asymptotes by assuming $y = xz$, and then by means of the equation expressing x and y in terms of z. If the same value of z which renders x and y infinite give a finite value for the intercepts of the tangents, then these determine the position of the asymptotes.

(16) Let $$ay^3 - bx^3 + c^2xy = 0$$
be the equation to the curve: then assuming $y = xz$, we find
$$x = \frac{c^2z}{b - az^3}, \quad y = \frac{c^2z^2}{b - az^3}.$$

Now x and y are both infinite when $z = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, and the intercept of the tangent on the axis of y is
$$y_0 = \frac{-c^2xy}{3ay^2 + c^2x} = \frac{-c^2z}{3az^2 + \frac{c^2}{x}},$$
which when $z = \left(\frac{b}{a}\right)^{\frac{1}{3}}$, and consequently $x = \infty$ becomes
$$y_0 = -\frac{c^2}{3a^{\frac{2}{3}}b^{\frac{1}{3}}},$$
and the equation to the asymptote is
$$y = \left(\frac{b}{a}\right)^{\frac{1}{3}}\left(x - \frac{c^2}{3a^{\frac{1}{3}}b^{\frac{2}{3}}}\right).$$

(17) If the equation to the curve be

$$y^4 - x^4 + 2bx^2y = 0,$$

we find by the same means the equations to two asymptotes to be

$$y = x - \frac{b}{2}, \quad \text{and} \quad y = -\left(x + \frac{b}{2}\right).$$

(18) Find the asymptotes of the curve

$$x^3 - ay(x - b) = 0.$$

As this equation can be put under the form

$$ay = \frac{x^3}{x - b},$$

the curve has a rectilinear asymptote in the ordinate at a distance b from the origin. It has also a parabolic asymptote, for we have

$$ay = x^2\left(1 - \frac{b}{x}\right)^{-1};$$

and therefore for the asymptote

$$ay = x^2 - bx + b^2;$$

$$\text{or} \quad ay - \tfrac{3}{4}b^2 = (x - \tfrac{1}{2}b)^2;$$

the equation to a common parabola, the latus rectum of which is a, and the axis of which is parallel to that of y.

(19) The curve whose equation is

$$a^2y^2 - 2b^3y - x^4 = 0,$$

has two parabolic asymptotes whose equations are

$$x^2 = a\left(y - \frac{b^2}{a}\right), \quad \text{and} \quad x^2 = a\left(\frac{b^2}{a} - y\right).$$

Their common axis is therefore the axis of y, and their latera recta are equal to a, but they are turned in opposite directions.

It sometimes happens that we obtain an equation for an asymptote with possible coefficients, though for large values of one variable in the equation to the curve, the other variable becomes impossible. This apparent anomaly has been explained by Mr Walton*, by availing himself of the general interpretation which may be given to the symbols in analytical geometry. The impossibility of one of the variables, when certain values are assigned to the other, may be interpreted as signifying that the curve for these values leaves the plane to which it is referred. Now when by assigning an indefinitely large value to the one variable, the other tends to become again possible and to assume the form of the equation to a straight line, as is the case when we find a possible rectilinear asymptote, this indicates that the curve tends to return to the plane of reference, and that at an infinite distance it will coincide with it in a line, the equation to which is that of the asymptote.

(20) As an example of a curve having a possible asymptote to an impossible branch let us take the equation,

$$x^4 (y - c)^2 = b^4 (a^2 - x^2).$$

When $x = 0$, $y = \infty$ and is possible, and therefore the axis of y is an asymptote: this is one of the ordinary kind. But if we put the equation under the form

$$(y - c)^2 = b^4 \frac{(a^2 - x^2)}{x^4},$$

it is easily seen that when $x = \infty$, $y = c$. On the other hand, if $x > a$, y is impossible. Hence the line whose equation is $y = c$ is an asymptote to an impossible branch of the curve; that is to say, a branch of the curve leaves the plane of reference when $x = \pm a$, but tends to return to it again when $x = \pm \infty$, coinciding then with the line whose equation is $y = c$. The form of the curve is given in fig. 27, where the dotted curve represents the impossible branches of the curve lying in a plane at right angles to the plane of the paper.

On the subject of asymptotes to curves, the reader may consult in addition to the work of Stirling before referred to, Newton's *Enumeratio Linearum Tertii ordinis*, and Cramer's *Analyse des Lignes Courbes*, Chap. VIII.

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* *Cambridge Mathematical Journal*, Vol. II. p. 236.

SECT. 2. *Polar Co-ordinates.*

If the curve be expressed by a relation between r and θ, then the tangent of the angle (ϕ) between the radius vector and the tangent to the curve is $r\dfrac{d\theta}{dr}$. The subtangent, which is the portion of a perpendicular to the radius vector at the origin intercepted by the tangent, is $r^2\dfrac{d\theta}{dr}$; and the perpendicular from the origin on the tangent is

$$p = \frac{r^2}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}}}.$$

If the curve be expressed by a relation between u and θ where $u = \dfrac{1}{r}$, the subtangent and perpendicular are equal to

$$-\frac{d\theta}{du} \text{ and } \frac{1}{\left\{u^2 + \left(\dfrac{du}{d\theta}\right)^2\right\}^{\frac{1}{2}}} \text{ respectively.}$$

Asymptotes to spirals are determined by finding what value of θ makes r infinite; and if the same value of θ make $r^2\dfrac{d\theta}{dr}$ either finite or equal to zero, a line drawn through the extremity of the subtangent parallel to r is an asymptote to the curve.

Spirals may have asymptotic circles: these are found by the condition that an infinite value of θ gives a finite value for r.

Ex. 1. The equation to the spiral of Archimedes is

$$r = a\theta.$$

The angle between the radius and tangent is

$$\phi = \tan^{-1} r\frac{d\theta}{dr} = \tan^{-1}\theta.$$

$$\text{The subtangent} = \frac{r^2}{a}.$$

The equation to the locus of the extremity of the subtangent is evidently

$$r' = \frac{r^2}{a} = a\theta^2,$$

θ being measured from a line 90° distant from the original axis as r' is at right angles to r. If in a similar way we find the locus of the extremity of the subtangent of the curve $r' = a\theta^2$, and so on in succession, we shall have a series of spirals, the equations to which are

$$r'' = \frac{a\theta^3}{1.2}, \quad r''' = \frac{a\theta^4}{1.2.3} \dots\dots r^{(n)} = \frac{a\theta^n}{1.2\dots(n-1)},$$

the angle θ in each case being measured from a line 90° distant from that in the preceding curve.

(2) The equation to the hyperbolic spiral is

$$r = \frac{a}{\theta}, \quad \text{or} \quad u = \frac{\theta}{a};$$

$$\text{therefore the subtangent} = \frac{d\theta}{du} = a.$$

The locus of the extremity of the subtangent is evidently a circle, the radius of which is a: and as $\theta = 0$ makes $r = \infty$ while the subtangent remains finite and equal to a, it appears that a line drawn parallel to the axis at a distance a is an asymptote.

(3) The equation to the lituus is

$$r = \frac{a}{\theta^{\frac{1}{2}}}, \quad \text{or} \quad u = \frac{\theta^{\frac{1}{2}}}{a};$$

$$\text{then} \quad \phi = \tan^{-1}(-2\theta), \quad \text{subtangent} = 2a\theta^{\frac{1}{2}};$$

and as $\theta = 0$ makes $r = \infty$ and the subtangent $= 0$, it appears that the line from which θ is measured is an asymptote to the curve.

Also since $r^2\theta = a^2$ it appears that if a circle be described with radius r, the sector between the axis and the radius r is of constant area.

(4) The equation to the Lemniscate is

$$r^2 = a^2 \cos 2\theta.$$

The perpendicular on the tangent is $\dfrac{r^3}{a^2}$:

$$\phi = \tan^{-1} r \frac{d\theta}{dr} = \tan^{-1}(-\cot 2\theta) = 2\theta - \frac{\pi}{2}.$$

(5) The equation to the logarithmic spiral is

$$r = c\epsilon^{\frac{\theta}{a}}.$$

Then $\phi = \tan^{-1} a$, and is therefore constant;

$$p = r \sin(\tan^{-1} a) = \frac{ra}{(1+a^2)^{\frac{1}{2}}}.$$

The subtangent $= ra$.

The locus of the extremity of the subtangent is the involute of the curve, the equation to it being

$$r_1 = ar = ac\epsilon^{\frac{\theta}{a}},$$

and therefore a similar spiral.

Also if r_2 be the subnormal, that is, the portion of a perpendicular to the radius vector at the origin cut off by the normal, the locus of the extremity of r_2 is the evolute of the spiral, its equation being

$$r_2 = \frac{r}{a} = \frac{c}{a}\epsilon^{\frac{\theta}{a}}.$$

(6) The equation to the Cardioid is

$$r = a(1 - \cos\theta).$$

If r' be a radius in the direction of r produced backwards,

$$r' = a\{1 - \cos(\theta + \pi)\} = a(1 + \cos\theta).$$

Therefore $r + r' = 2a$, or the chords passing through the pole are of constant length.

$$\tan\phi = \tan\tfrac{1}{2}\theta; \text{ therefore } \phi = \tfrac{1}{2}\theta.$$

(7) Let the equation to the spiral be

$$r^n = a^n \sin n\theta.$$

Then $\qquad \tan\phi = \tan n\theta$; and $\phi = n\theta$.

If ϕ_1 be the value of ϕ corresponding to an angle $\theta + \pi$; that is, to a tangent at the other extremity of the chord passing through the origin, $\phi_1 = n(\theta + \pi)$ and $\phi_1 - \phi = n\pi$. Therefore the angle between two tangents at the extremities of any chord passing through the origin is constant.

(8) Let the equation to a spiral be

$$\theta(2ar - r^2)^{\frac{1}{2}} = 1.$$

Then when $\theta = \infty$, $(2ar - r^2)^{\frac{1}{2}} = 0$ and $r = 0$, $r = 2a$.

Therefore the circle, the radius of which is $2a$, is an asymptote to the spiral. The pole also, for which $r = 0$, may also be considered as an asymptotic circle the radius of which is zero, as the curve makes an infinite number of revolutions before it reaches it. The same remark applies to the logarithmic spiral, and many other curves for which r is zero when θ is infinite.

(9) The curve whose equation is

$$r = \frac{a\theta^2}{\theta^2 - 1},$$

offers examples of both rectilinear and circular asymptotes.

For if $\theta = \pm 1$, $r = \infty$, and as $r^2 \dfrac{d\theta}{dr} = -\dfrac{a\theta^3}{2}$, the subtangent corresponding to $\theta = \pm 1$ is $\mp \frac{1}{2}a$, and there are therefore two rectilinear asymptotes inclined at angles $+1$ and -1 to the axis.

Also since $r = \dfrac{a\theta^2}{\theta^2 - 1} = a\left(1 - \dfrac{1}{\theta^2}\right)^{-1} = a$ when $\theta = \infty$, the circle whose radius is a is asymptotic to the spiral.

The form of the curve is given in fig. 28.

The perpendicular on the tangent is $\frac{r^3}{a^2}$:

$$\phi = \tan^{-1} r\frac{d\theta}{dr} = \tan^{-1}(-\cot 2\theta) = 2\theta - \frac{\pi}{2}.$$

(5) The equation to the logarithmic spiral is

$$r = c\epsilon^{\frac{\theta}{a}}.$$

Then $\phi = \tan^{-1} a$, and is therefore constant;

$$p = r\sin(\tan^{-1} a) = \frac{ra}{(1+a^2)^{\frac{1}{2}}}.$$

The subtangent $= ra$.

The locus of the extremity of the subtangent is the involute of the curve, the equation to it being

$$r_1 = ar = ac\epsilon^{\frac{\theta}{a}},$$

and therefore a similar spiral.

Also if r_2 be the subnormal, that is, the portion of a perpendicular to the radius vector at the origin cut off by the normal, the locus of the extremity of r_2 is the evolute of the spiral, its equation being

$$r_2 = \frac{r}{a} = \frac{c}{a}\epsilon^{\frac{\theta}{a}}.$$

(6) The equation to the Cardioid is

$$r = a(1 - \cos\theta).$$

If r' be a radius in the direction of r produced backwards,

$$r' = a\{1 - \cos(\theta + \pi)\} = a(1 + \cos\theta).$$

Therefore $r + r' = 2a$, or the chords passing through the pole are of constant length.

$$\tan\phi = \tan\tfrac{1}{2}\theta;\ \text{therefore}\ \phi = \tfrac{1}{2}\theta.$$

(7) Let the equation to the spiral be

$$r^n = a^n \sin n\theta.$$

Then $\tan\phi = \tan n\theta$; and $\phi = n\theta$.

If ϕ_1 be the value of ϕ corresponding to an angle $\theta + \pi$; that is, to a tangent at the other extremity of the chord passing odd order, then $\frac{dy}{dx}$ is a maximum or minimum, and the curve has a point of contrary flexure. Instead of finding what differential coefficient vanishes, it is generally more convenient to try whether $\frac{d^2y}{dx^2}$ change sign on substituting in it values of x a little greater and a little less than that which makes it vanish. If it do change sign, the point is one of contrary flexure, otherwise not. If $\frac{d^2y}{dx^2} = \infty$, there may be a point of contrary flexure provided that it change sign for values of x a little greater or a little less than that which makes it infinite.

If any values of x and y make $\frac{dy}{dx} = \frac{0}{0}$, it is an indication generally that the point in question is a multiple point, or that several branches of the curve pass through it. The multiplicity may be of different kinds. 1st. If $\frac{dy}{dx}$ is found by the usual method of evaluating vanishing fractions, to have several different possible values there are as many branches of the curve cutting each other in one point. 2nd. If $\frac{dy}{dx}$ is found to have two or more equal and possible values, there are two or more branches of the curve touching each other in one point, which is called a point of *osculation.* 3rd. If all the values of $\frac{dy}{dx}$ are found to be impossible, then the point in question is an isolated or conjugate point, that is, one through which there passes no branch in the plane of the co-ordinate axes. In fact the point is that in which impossible branches of the curve meet the plane of the axes. With respect to the 2nd and 3rd class of multiple points a few more remarks are necessary. If when $\frac{dy}{dx}$ has two equal values for a given value a of one of the variables, we find that for a value $a+h$ the other variable is possible, and for a value $a-h$ impossible, or *vice versâ*, the curve stops short at the point in question, and is doubled back on itself, forming what is called a *cusp*. The cusp is said to be of the *first* species or a *ceratoid** when the branches touch the common tangent on opposite sides; and of the *second* species or a *ramphoid*† when they touch on the same side. These may be distinguished by the consideration that in the first the values of $\frac{d^2y}{dx^2}$ are of opposite, and in the second of the same signs. It is to be observed that at a cusp the two branches of the curve never make with each other an angle the trigonometrical tangent of which is of finite magnitude: we cannot properly say that the angle itself is infinitely small, as in fact it is equal to two right angles, the inclination of the one branch of the curve being measured in a direction opposite to that of the other. Although the condition of $\frac{dy}{dx}$ when of the form $\frac{0}{0}$ having impossible values always indicates a conjugate point, yet it may happen that $\frac{dy}{dx}$ and any number of the differential coefficients are possible at a conjugate point. In such cases the impossible branch of the curve does not pierce the plane of the axes, but touches it at the conjugate point, the order of contact being that of the highest differential coefficient which is possible. To determine with certainty whether a point be or be not a conjugate point or a cusp, it is always necessary to try whether the equation to the curve gives possible values for both variables on each side of the point in question.

If some of the values of $\frac{dy}{dx}$ be possible and some impossible for the given value of x, there is a conjugate point situate on the curve; that is, a branch in the impossible plane pierces the plane of reference in a point through which there passes a possible branch of the curve.

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* Κέρας, a horn.

† Ῥάμφος, a beak.

For a fuller development of the relation between the various kinds of points indicated by the condition $\frac{dy}{dx}=\frac{0}{0}$, the reader is referred to a paper by Mr Walton in the *Cambridge Mathematical Journal*, Vol. II. p. 155.

If the equation to the curve be put under the more symmetrical form

$$u=f(x, y)=0,$$

we easily obtain analytical conditions for distinguishing between the three classes of double points indicated by the condition $\frac{dy}{dx}=\frac{0}{0}$, viz. true double points, points of osculation, and conjugate points. The condition $\frac{dy}{dx}=\frac{0}{0}$ involves the two,

$$\frac{du}{dx}=0, \quad \frac{du}{dy}=0.$$

Proceeding to the differential of the second order, we find in consequence of the preceding condition

$$\frac{d^2u}{dx^2}+2\frac{d^2u}{dx\,dy}\frac{dy}{dx}+\frac{d^2u}{dy^2}\left(\frac{dy}{dx}\right)^2=0,$$

whence we find

$$\frac{dy}{dx}=\frac{-\frac{d^2u}{dx\,dy}\pm\left\{\left(\frac{d^2u}{dx\,dy}\right)^2-\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right)\right\}^{\frac{1}{2}}}{\frac{d^2u}{dy^2}}.$$

Now for a true double point we must have two possible values for $\frac{dy}{dx}$; for a point of osculation we must have the two values equal; and for a conjugate point we must have the two values impossible. Hence we have the three conditions:

$$\left(\frac{d^2u}{dx\,dy}\right)^2-\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right)>0 \text{ for a true double point,}$$

$$\left(\frac{d^2u}{dx\,dy}\right)^2 - \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right) = 0 \text{ for a point of osculation,}$$

$$\left(\frac{d^2u}{dx\,dy}\right)^2 - \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right) < 0 \text{ for a conjugate point.}$$

If the point be more than double, it is necessary to proceed to higher differentiations, but the formulæ become too complicated to be of much use.

The second of the preceding conditions furnishes an easy demonstration of the following general property of curves of the third order. "The three asymptotes of a curve of the third order being given, the locus of the points of osculation is the maximum ellipse which can be inscribed in the triangle formed by the asymptotes: the locus of the conjugate points is within, and of the double points without this ellipse."

If we refer a curve of the third order to two of its asymptotes as axes, their intersection being the origin, its equation must evidently be of the form,

$$ax^2y + 2bxy + cxy^2 = h.$$

Hence $\dfrac{du}{dx} = 2axy + 2by + cy^2$,

$$\frac{du}{dy} = ax^2 + 2bx + 2cxy,$$

$$\frac{d^2u}{dx^2} = 2ay, \quad \frac{d^2u}{dy^2} = 2cx, \quad \frac{d^2u}{dx\,dy} = 2\,(ax + b + cy).$$

Therefore by the condition for a point of osculation

$$(ax + b + cy)^2 - acxy = 0,$$

$$\text{or } a^2x^2 + acxy + c^2y^2 + 2abx + 2bcy + b^2 = 0,$$

which is the equation to an ellipse.

That this ellipse is the maximum ellipse inscribed in the triangle formed by the asymptotes is easily shown. The equations to the three asymptotes are

$$x = 0, \; y = 0, \text{ and } ax + cy + 2b = 0.$$

From the last it appears that the intercepts of the axes cut off by the third asymptote are $-\dfrac{2b}{a}$ and $-\dfrac{2c}{a}$. Also from the equation to the ellipse it appears that it touches the axes at distances $-\dfrac{b}{a}$ and $-\dfrac{c}{a}$ from the origin, or that the points of contact bisect these two sides of the triangle. If in the value of $\dfrac{dy}{dx}$ derived from the equation to the ellipse we substitute the values $-\dfrac{b}{a}$ and $-\dfrac{b}{c}$ for x and y, we find $\dfrac{dy}{dx} = -\dfrac{a}{c}$, which is the same as that derived from the equation to the third asymptote, and as these values of x and y satisfy both the equation to the ellipse and that to the asymptote, it appears that the ellipse touches all the three sides of the triangle in their middle points, which by Chap. VII. Ex. 19, is the property of the maximum ellipse. The latter part of the theorem is too obvious to need demonstration. This proposition is due to Plucker, *Journal de Mathématiques*, (*Liouville*) Vol. II. p. 11.

Points of Contrary Flexure or of Inflexion.

Ex. 1. The equation to the Witch of Agnesi is

$$xy = 2a\,(2ax - x^2)^{\frac{1}{2}};$$

whence we find

$$\frac{d^2y}{dx^2} = \frac{2a^2\,(3a - 2x)}{x\,(2ax - x^2)^{\frac{3}{2}}};$$

$\dfrac{d^2y}{dx^2} = 0$ gives $x = \dfrac{3a}{2}$ and $y = \pm\dfrac{2a}{3^{\frac{1}{2}}}$, and as $\dfrac{3a}{2} + h$ and $\dfrac{3a}{2} - h$, when substituted for x, make $\dfrac{d^2y}{dx^2}$ change sign, there are two points of contrary flexure corresponding to these values of x and y.

$\frac{d^2y}{dx^2}$ and $\frac{dy}{dx}$ both become infinite when $x = 0$ and when $x = 2a$, but neither of those values gives a point of inflexion, since y is impossible when x is negative or greater than $2a$.

(2) The curve whose equation is

$$x^3 - 3bx^2 + a^2y = 0$$

has a point of inflexion the co-ordinates of which are

$$x = b, \quad y = \frac{2b^3}{a^2}.$$

(3) Let the equation to the curve be

$$ax^3 + by^3 - c^4 = 0.$$

There are two points of inflexion, the co-ordinates of the one being $x = 0, \quad y = c\left(\frac{c}{b}\right)^{\frac{1}{3}}$,

those of the other $x = c\left(\frac{c}{a}\right)^{\frac{1}{3}}, \quad y = 0.$

(4) Let the equation to the curve be

$$x^4 - a^2x^2 + a^3y = 0.$$

There are two points of inflexion corresponding to

$$x = \pm\frac{a}{6^{\frac{1}{2}}}, \quad y = \frac{5a}{36}.$$

(5) Let the equation to the curve be

$$y = b + (x - a)^{\frac{m}{n}},$$

where m and n are both odd.

If $\frac{m}{n} > 1$, $x = a$ gives a point of inflexion, the tangent being parallel to the axis of x.

If $\frac{m}{n} < 1$, $x = a$ gives a point of inflexion corresponding to $\frac{d^2y}{dx^2} = \infty$, the tangent being perpendicular to the axis of x.

(6) In the curve of sines the equation to which is

$$y = c\sin\frac{x}{a},$$

there is a point of inflexion wherever the curve cuts the axis of x.

In polar curves points of inflexion are found by the conditions that at such points $\frac{dp}{dr} = 0$, and changes sign in passing through zero.

(7) In the lituus $r^2 = \frac{a^2}{\theta}$; whence we find

$$p = \frac{2a^2 r}{(r^4 + 4a^4)^{\frac{1}{2}}}.$$

When $r = \pm a 2^{\frac{1}{2}}$ or $\theta = \frac{1}{2}$, $\frac{dp}{dr} = 0$, and changes sign on either side of the point corresponding to these values: the point is therefore one of inflexion.

(8) In the Lemniscate of Bernoulli

$$r^2 = a^2\cos 2\theta,$$

$$\text{and } p = \frac{r^3}{a^2}, \qquad \frac{dp}{dr} = \frac{3r^2}{a^2}.$$

Hence the origin is a point of inflexion for two branches of the curve.

(9) The equations to the Trochoid are

$$x = a(\theta - e\sin\theta), \quad y = a(1 - e\cos\theta),$$

whence we find

$$\frac{d^2y}{dx^2} = \frac{e(\cos\theta - e)}{(1 - e\cos\theta)^3} = 0;$$

therefore when $\cos\theta = e$ and $y = a(1 - e^2)$ there is a point of inflexion.

The preceding examples are taken chiefly from Cramer, *Analyse des Lignes Courbes*, Chap. XI.

Multiple Points.

Among these I include all those points for which we find $\frac{dy}{dx} = \frac{0}{0}$, including points where several branches intersect, or nodes, points of osculation, cusps, and conjugate points.

(1) Let the equation to the curve be

$$ay^2 - x^3 - bx^2 = 0.$$

Here $\frac{dy}{dx} = \frac{3x^2 + 2bx}{2ay} = \frac{0}{0}$ when $x = 0$, $y = 0$. By the usual method of evaluating vanishing fractions we find

$$\frac{dy}{dx} = \frac{6x + 2b}{2a\frac{dy}{dx}};$$

whence $\left(\frac{dy}{dx}\right)^2 = \frac{6x + 2b}{2a} = \frac{b}{a}$, when $x = 0$.

Therefore $\frac{dy}{dx} = \pm \left(\frac{b}{a}\right)^{\frac{1}{2}}$, indicating that at the origin two branches of the curve intersect, making with the axis of x angles the tangents of which are $\left(\frac{b}{a}\right)^{\frac{1}{2}}$ and $-\left(\frac{b}{a}\right)^{\frac{1}{2}}$ respectively.

(2) The curve

$$x^4 - ayx^2 + by^3 = 0$$

has at the origin a triple point formed by the intersection of three branches of the curve. The values of $\frac{dy}{dx}$ at the point are $\pm \left(\frac{b}{a}\right)^{\frac{1}{2}}$ and 0. See fig. 29.

(3) The curve

$$x^4 - 2ax^2y - 2x^2y^2 + ay^3 + y^4 = 0$$

has at the origin a triple point, the values of $\frac{dy}{dx}$ being $\pm \frac{1}{2}$ and 0. The form of the curve is given is fig. 30.

(4) Let the curve be

$$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$

Here $\dfrac{du}{dx} = 4(x^3 - a^2x)$, $\qquad \dfrac{du}{dy} = -6(ay^2 + a^2y)$.

Both of these vanish when $y = 0$ and $x = \pm a$, and when $y = -a$ and $x = 0$. There are three double points corresponding to these values of x and y.

$$\text{For } y = 0, \qquad x = +a, \qquad \frac{dy}{dx} = \pm\left(\frac{4}{3}\right)^{\frac{1}{2}},$$

$$y = 0, \qquad x = -a, \qquad \frac{dy}{dx} = \pm\left(\frac{4}{3}\right)^{\frac{1}{2}},$$

$$y = -a, \qquad x = 0, \qquad \frac{dy}{dx} = \pm\left(\frac{2}{3}\right)^{\frac{1}{2}}.$$

For the form of the curve see fig. 31.

(5) In the curve

$$x^4 + x^2y^2 - 6ax^2y + a^2y^2 = 0$$

we find $\left(\dfrac{dy}{dx}\right)^2 = 0$ at the origin, or two branches there touch each other as in fig. 32.

(6) In the curve

$$x^5 + bx^4 - a^3y^2 = 0$$

we find at the origin

$$\left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2u}{dy^2}\right) - \left(\frac{d^2u}{dx\,dy}\right)^2 = 0,$$

which indicates a point of osculation, and as $\dfrac{dy}{dx} = 0$ at the origin, the two branches touch the axis of x. See fig. 33.

(7) The curve

$$(by - cx)^2 = (x - a)^5$$

has a cusp of the first species when $x = a$; the common tangent is parallel to the axis of x. See fig. 34.

(8) The curve

$$x^4 - ax^2y - axy^2 + a^2y^2 = 0$$

has at the origin a ramphoid cusp, the axis of x being the common tangent. See fig. 35.

(9) The curve

$$y^4 - axy^2 + x^4 = 0$$

has at the origin a ceratoid cusp touching the axis of x, and also a branch touching the axis of y. See fig. 36.

(10) The curve

$$ay^2 - x^3 + bx^2 = 0$$

has a conjugate point at the origin, since $x = 0$, $y = 0$ satisfy the equation, but $x = \pm h$ when h is small make y impossible. At the origin $\frac{dy}{dx}$ takes the form $\frac{0}{0}$, and its true value is $\left(-\frac{b}{a}\right)^{\frac{1}{2}}$, which indicates that there are two impossible branches passing through the plane of the axes at the origin.

(11) The curve whose equation is

$$(c^2y - x^3)^2 = (x - b)^5 (x - a)^6, \qquad a < b,$$

has a conjugate point whose co-ordinates are

$$x = a, \quad y = \frac{a^3}{c^2},$$

but the differential coefficients are possible till we come to the third, showing that the impossible branch has a contact of the second order with the plane of the axes.

(12) In the curve

$$a^3y^2 - 2abx^2y - x^5 = 0$$

there is a point of osculation at the origin, and one of the branches experiences an inflexion. Such a point is called one of oscul-inflexion. See fig. 37.

(13) The curve

$$y^5 + ax^4 - b^2xy^2 = 0$$

has a ceratoid cusp at the origin and an inflexion in another branch at the same point. The cusp has the axis of x as tangent, and the inflected branch touches the axis of y. The form of the curve is that of the letter R. See fig. 38.

(14) The curve

$$(y - c)^2 = (x - a)^6 (x - b), \quad a > b,$$

has an oval between $x = a$ and $x = b$. When $x = a$ and $y = c$ there is a point of osculation, the common tangent being parallel to the axis of x. See fig. 39.

(15) The curve

$$(x^2 + y^2)^3 = 4a^2x^2y^2$$

has at the origin a quadruple point, a pair of branches touching both the axes. The form of the curve is best seen by transferring the equation to polar co-ordinates, when it becomes $r = a \sin 2\theta$.

The greater number of the preceding examples are taken from Cramer's work, Chap. x. and Chap. XIII.

CHAPTER XI.

ON THE TRACING OF CURVES FROM THEIR EQUATIONS.

SECT. 1. *Curves referred to Rectangular Co-ordinates.*

BEFORE proceeding to give examples of the application of analysis to determine the form of curves when their equations are given, I shall say a few words on the principles of the interpretation of symbols in analytical geometry, as a knowledge of these is requisite for the understanding of the views which I have adopted both in the preceding and in the following pages.

By the principles of the Geometry of Descartes, the position of a point in a plane is known when its distances from two axes Ox, Oy intersecting each other at right angles are known: and a curve is defined as a series of points for which there exists the same relation between the ordinate y and the abscissa x. This relation is expressed by means of an equation $f(x, y) = 0$ between x, y and constants, which is called the equation to the curve. If we assign a series of values to one of the two variables x and y, the corresponding values of the other can be found by means of the equation $f(x, y) = 0$: now so long as we consider this only as an arithmetical equation, the only values of x and y which we can use are positive numbers. If we agree that the values of x are to represent lines measured from O (fig. 40) along Ox, and values of y lines measured from O along Oy, we can by means of the arithmetical values alone of x and y determine the positions of all points within the angle xOy. But the equation $f(x, y) = 0$ for any value of one variable will frequently give an expression for the other variable which is not arithmetical, such as $-a$ or $(-a^2)^{\frac{1}{2}}$, or more generally $(+a^n)^{\frac{1}{n}}$. Now there is no *necessity* for interpreting these expressions which are uninterpretable in arithmetic; but it is clear that we shall gain an advantage in the generalization of our results if we are able to interpret these expressions in any way consistent with the original definition of the symbols employed. It was soon seen by the early cultivators of this geometry that the first of these expressions $(-a)$ could receive the geometrical interpretation that, if a represented a line measured in one direction, $(-a)$ represented the same length of line measured in the opposite direction. This extension of the interpretation of the symbols is of great importance, since it enables us to express by the one equation, $f(x, y) = 0$, the position of a point in all parts of the plane in which the axes Ox and Oy lie; and no curve is considered to be completely traced unless the negative, as well as the positive, values of the variables be taken into account. This however is merely a matter of convention, and we might, if it were thought proper, restrict ourselves to the positive values of the variables and confine the curve to the angle xOy. If instead of interpreting $(-a)$ to mean the measuring of the length a in a direction opposite to that originally taken, we use the more general definition that $-a$ means that the line a is to be turned round through two right angles, we are led to the general interpretation of such an expression as $(+a^n)^{\frac{1}{n}}$, viz. that the line a is to be turned round through the nth part of four right angles. This gives us a farther extension of the use of the equation $f(x, y) = 0$; for, as the turning of a line through a given angle is not confined to any one plane, we are enabled to express by the equation to the curve the position of a point situate in any part of space. To explain this, let us suppose that for a value $x = a$, we obtain a value $y = (+)^{\frac{m}{n}}b$; this implies that the length b is to be measured not along the axis of y, but along a line inclined to it at an angle $\frac{m}{n}2\pi$: but as the axes are supposed to remain perpendicular to each other, this angle must be taken in a plane perpendicular to that of the original axes. Hence, if there be a series of values of y all affected by the same quantity $(+)^{\frac{m}{n}}$, they will give rise to a branch of the curve lying in a plane inclined at an angle $\frac{m}{n}2\pi$ to the plane of the original axes. If for different values of x the index of + change its value, the branch does not lie in one plane, but is a curve of double curvature.

This use of the interpretation of the symbol $(+a^n)^{\frac{1}{n}}$ has not been generally adopted, but it is quite as legitimate an extension as that of the negative values of the variables, and for the thorough understanding of the course of a curve it is quite as necessary. For all the ordinary purposes however of the equations to curves it is sufficient to use only the positive and negative values of the variables, and to these I shall restrict myself, only observing, that when such an expression as $(-a^2)^{\frac{1}{2}}$ occurs, it is not to be called imaginary, nor is the curve to be said therefore to have no existence for that value; but it is to be interpreted as indicating that the curve there leaves the plane of the axes, which for convenience I shall call the plane of reference.

The student who wishes for more information regarding the general interpretation of formulæ in Analytical Geometry is referred to a paper by the Abbé Buée in the *Philosophical Transactions* for 1806, to Mr Warren's *Tract on the Geometrical Interpretation of Imaginary Quantities*, and to the *Cambridge Mathematical Journal*, Vol. I. p. 259, and Vol. II. p. 103 and p. 155: the last two papers being by Mr Walton.

When we proceed to trace a curve from its equation it is advisable in the first place to solve the equation with respect to one or other of the variables, if the solution be in a form which enables us to determine readily its value for different values of the other variable. After that we may proceed in the following way.

1. If y be the variable which is expressed in terms of x, assign to x all positive values from 0 to ∞, marking those which make $y=0$, $y=\infty$, or y impossible. The first gives the points where the curve cuts the axis of x, the

second gives the infinite branches, and the third, showing where the curve quits the plane of reference, gives the limits of the curve in that plane.

2. Assign to x all negative values from 0 to ∞, proceeding as in the case of the positive values of x. In both cases attend to both the positive and negative values of y, so as to obtain the branches on both sides of the line of abscissæ.

3. Find whether the curve have asymptotes, and determine them if they exist.

4. Find the value of $\frac{dy}{dx}$, and thence deduce the maximum and minimum points of the curve, and the angles at which the curve cuts the axes.

5. Find the value of $\frac{d^2y}{dx^2}$, and thence deduce the nature of the curvature of the different branches, and the points of contrary flexure if such exist.

6. Determine the existence and nature of the singular points by the usual rules.

Ex. 1. Let the equation to be discussed be

$$y^2 = \frac{x^3 - a^3}{x + b}.$$

From its form we see at once that there are always for each value of x two values of y equal but of opposite signs; hence the curve is symmetrical with regard to the axis of x.

Let x be *positive;* when x is between 0 and a, y is impossible, and the curve does not exist in the plane of reference: when $x = a$, $y = 0$: when $x > a$, y is possible, and increases without limit as x so increases.

Let x be *negative;* when x is between 0 and b, y is impossible, and there is no branch in the plane of reference: when $x = b$, y is infinite: when $x > b$, y increases without limit as x so increases. Hence it appears that the curve has six infinite branches.

Since $x = -b$ makes y infinite, the ordinate at that point is an asymptote. Also since

$$y = \pm \frac{(x^3 - a^3)^{\frac{1}{2}}}{(x + b)^{\frac{1}{2}}} = \pm x \left(1 - \frac{a^3}{x^3}\right)^{\frac{1}{2}} \left(1 + \frac{b}{x}\right)^{-\frac{1}{2}};$$

on expanding, and neglecting negative powers of x, we find

$$y = \pm (x - \tfrac{1}{2} b)$$

as the equation to two asymptotes inclined at angles $+ 45^0$ and $- 45^0$ to the axis of x.

On combining the equation of this asymptote with that of the curve, we find that there is a value of x corresponding to an intersection of the curve with the asymptote.

Differentiating the equation to the curve, we find

$$2 \frac{dy}{dx} = \frac{2x^3 + 3bx^2 + a^3}{(x^3 - a^3)^{\frac{1}{2}} (x + b)^{\frac{3}{2}}}.$$

This equated to zero gives a cubic equation, which must have one real root negative, since all the terms of the numerator are positive: this indicates a minimum ordinate. The course of the curve shows that the other two roots of the cubic must be impossible.

When $x = a$, $\dfrac{dy}{dx}$ is infinite, or the curve cuts the axis at right angles.

The value of $\dfrac{d^2y}{dx^2}$ shows that the curve is always concave to the axis of x when x is positive, and convex when it is negative.

The form of the curve is given in fig. 43, where $OA = a$, $OB = b$; ON is the abscissa corresponding to the intersection of the curve with the asymptote; and OM is the abscissa of the minimum ordinate.

(2) Let the equation to the curve be

$$y^3 = \frac{x^4 - a^2x^2}{2x - a}.$$

This curve, see fig. 44, has four infinite branches, and the equations to its asymptotes are

$$x = \frac{a}{2}, \qquad y = \frac{1}{2^{\frac{1}{3}}} \left(x + \frac{a}{6}\right).$$

The curve cuts the axis of x at right angles at the origin, and at distances $+a$ and $-a$ from the origin: at the latter two points there are points of contrary flexure, while the origin is a cusp. There is a maximum value of y corresponding to a value of x between 0 and $-a$.

(3) $$xy^2 + 2a^2y - x^3 = 0.$$

Solving the equation with respect to y, we find

$$y = -\frac{a^2}{x} \pm \left(x^2 + \frac{a^2}{x^2}\right)^{\frac{1}{2}}.$$

When $x = 0$, $y = 0$, and $y = -\infty$. This will be readily seen by putting the original equation under the form

$$y\left(y + \frac{2a^2}{x}\right) - x^3 = 0,$$

which when $x = 0$ gives $y = 0$, and $y + \frac{2a^2}{0} = 0$ or $y = -\infty$.

To determine the effect of increasing x positively, let us consider the two values of y separately. Taking the upper sign and expanding the radical in ascending powers of x, we have

$$y = -\frac{a^2}{x} + \frac{a^2}{x}\left(1 + \frac{1}{2}\frac{x^4}{a^4} - \frac{1.1}{2^2.1.2}\frac{x^8}{a^8} + \&c.\right),$$

$$\text{or } y = \frac{1}{2}\frac{x^3}{a^3} - \frac{1.1}{2^2.1.2}\frac{x^7}{a^6} + \&c.$$

Now when x is small, the first term gives the sign to the series, and y is therefore positive; and as no value of x can make $y = 0$, this branch of the curve lies always in the first quadrant, and extends to infinity, since $y = \infty$, when $x = \infty$.

Taking the lower sign and expanding the radical in descending powers of x, we have

$$y = -\frac{a^2}{x} - x\left(1 + \frac{1}{2}\frac{a^4}{x^4} - \&c.\right),$$

which when $x = \infty$ is negative and infinite: expanding in ascending powers of x, we have

$$y = -\frac{2a^2}{x} - \left(\frac{1}{2}\frac{x^3}{a^2} - \frac{1.1}{2^2.1.2}\frac{x^7}{a^6}\ \&c.\right),$$

which when $x = 0$ is negative and infinite; hence this branch lies wholly in the fourth quadrant.

For the negative values of x it is sufficient to observe that as the original equation remains unchanged when $-x$ and $-y$ are substituted for $+x$ and $+y$, it follows that the opposite quadrants are symmetrical, and we need therefore only investigate the form of the curve in the first and fourth quadrants.

To determine the asymptotes: since $y = -\infty$ when $x = 0$, the axis of y is an asymptote to the branch in the fourth quadrant: also by expanding the value of y in descending powers of x we have, neglecting the terms involving negative powers of x,

$$y = \pm x,$$

as the equations to two other asymptotes.

Differentiating the value of y, we find that at the origin $\frac{dy}{dx} = 0$, and therefore that the curve then touches the axis of x. We also find a minimum value for y when $x = \pm 3^{\frac{1}{4}} a$. This minimum value of y belongs only to the branches in the second and fourth quadrants, and not to the branches in the first and third quadrants.

Without proceeding to find the value of $\frac{d^2y}{dx^2}$, it is not difficult to see that at the origin there is a point of contrary flexure, since the curve there both touches and cuts the axis of x. The form of the curve is given in Fig. (45).

When the equation cannot be solved with respect to one or other of the variables, it is necessary to have recourse to particular artifices suited to the case under consideration.

(4) Let the equation to be discussed be

$$x^3 - 3axy + y^3 = 0.$$

When $x = 0$, $y^3 = 0$: the multiplicity of values of y shows that there is a multiple point at the origin. Differentiating, we have

$$\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax} = \frac{0}{0} \text{ when } x = 0, \quad y = 0.$$

To find the true value of this fraction, differentiate its numerator and denominator; then

$$\frac{dy}{dx} = \frac{a\dfrac{dy}{dx} - 2x}{2y\dfrac{dy}{dx} - a};$$

therefore when $x = 0$,

$$2\left\{y\left(\frac{dy}{dx}\right)^2 - a\frac{dy}{dx}\right\} = 0,$$

which as $y = 0$ when $x = 0$, gives

$$\frac{dy}{dx} = 0; \quad \frac{dy}{dx} = \infty;$$

therefore at the origin one branch touches the axis of x and the other that of y.

To find the points where the tangent is parallel to the axis of x make $\dfrac{dy}{dx} = 0$, whence $ay = x^2$; substituting this value in the equation to the curve, it becomes

$$x^6 - 2a^3x^3 = 0;$$

$$\text{whence } x = 0, \qquad x^3 = 2a^3.$$

The former value gives the origin; the latter gives one possible value $x = 2^{\frac{1}{3}}a$, to which corresponds $y = 2^{\frac{2}{3}}a$. From the symmetry of the equation it is easy to see that the curve is parallel to the axis of y when $y = 2^{\frac{1}{3}}a$ and $x = 2^{\frac{2}{3}}a$. Hence it appears that in the first quadrant there is a closed curve forming a loop which at the origin touches the two axes.

To find the asymptotes put $y = zx$, then we have

$$x = \frac{3az}{z^3 + 1}, \qquad y = \frac{3az^2}{z^3 + 1}.$$

When $z = -1$ both x and y are infinite. The expression for the intercept of the tangent on the axis of x is

$$\frac{-3az}{z^3 - 2} = -a \quad \text{when } z = -1.$$

Therefore a line inclined at angle of 135° to the axis of x, and cutting it at a distance $-a$ from the origin, is an asymptote to the curve. For the form of this curve, see (fig. 51).

(5) The form of the curve whose equation is

$$(x+b)y^2 = (x+a)x^2, \quad b > a,$$

is given in (fig. 47), where $OB = b$, $OA = a$.

The reader will find a great variety of curves discussed in the work of Cramer, before referred to. For lines of the third order he may consult Newton's *Enumeratio Linearum Tertii Ordinis*, and Stirling's *Commentary* on that work.

SECT. 2. *Curves referred to Polar Co-ordinates.*

When the equation to a curve is given by an equation

$$r = f(\theta),$$

a fixed point is to be taken as origin, and a fixed line passing through it as the axis from which θ is to be measured. The values of θ which make $f(\theta) = 0$ are then to be found; these give the angles at which the branches of the curve which pass through the origin cut the axis. By giving to θ the values 0 and $n\pi$ we find the values of r when the curve cuts the axis; and by giving to θ the value $\frac{1}{2}(2n+1)\pi$ we find the values of r when the radius is perpendicular to the axis. By making $\frac{dr}{d\theta} = 0$ we find the values of θ, for which r is a maximum or minimum. After determining these points in the curve, the asymptotes, both rectilinear and circular, are to be sought out; and when these are known there will generally be little difficulty in finding the form of the curve, except when singular points occur; and these are to be investigated by the usual process.

It is to be observed that in all cases we must substitute both positive and negative values of θ, and that when the result gives a negative value for r, it is to be measured along the radius vector produced backwards: if this be not attended to, the curve will want branches or spires, and will appear to be discontinuous. Some authors neglect the negative values of r, and trace the spiral only with the positive values of the radius vector; that this is an incomplete mode of tracing the curve may easily be seen by transferring the equation from polar to rectilinear co-ordinates, when it will be found that, according to the principles of interpretation used for the latter, the tracing of the curve from its rectilinear equation will give more branches than that from the polar equation. The remark which was made regarding the interpretation of the symbols in rectilinear co-ordinates applies equally to polar: there is no necessity for interpreting all the symbols which arise in our operations, but we gain much in the generality of our formulæ when we do interpret them, and we should sacrifice many advantages by not doing so*.

Ex. (1) Let the equation to the curve be

$$r = a\cos\theta + b, \quad a > b.$$

When $\theta = 0$, $r = a + b$, a maximum.

From $\theta = 0$ to $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$, which is an angle in the second quadrant, r is positive and continually diminishing till when $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$ it is equal to 0, and therefore the curve passes through the pole cutting the axis at an angle whose cosine is $-\frac{b}{a}$.

From $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$ to $\theta = \pi$, r is negative and increasing, and being measured on the radius vector produced backwards it traces out the portion OEB (fig. 42) of the curve; and when $\theta = \pi$, $r = -(a - b) = OB$.

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* It has been usual among writers on this subject to neglect the negative values of r and so to deprive the curves of their due allowance of branches: a marked instance of this may be seen in the spiral of Archimedes, which, as usually traced, appears shorn of one half of its length. Professor De Morgan is, so far as I know, the only writer who has insisted on the interpretation of negative values of r. See his *Diff. Calc.* p. 342.

From $\theta = \pi$ to $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$ in the third quadrant r is still negative and diminishing, and traces out the portion *BFO* of the curve.

When $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$, $r = 0$, and the curve passes again through the pole, cutting the axis at the same angle as before, but measured in the opposite direction.

From $\theta = \cos^{-1}\left(-\frac{b}{a}\right)$ in the third quadrant to $\theta = 2\pi$, r is positive and increasing, till it again reaches the maximum value $a + b$ or *OA*, after tracing the portion *OGHA* of the curve. On increasing the values of θ the same values of r recur, showing that the curve is complete; and it is obviously unnecessary to give to θ negative values, since these will give the same values for r as the positive values $2\pi - \theta$ have done.

When $a = b$ the smaller oval *OEBF* vanishes, and the point *O* is a cusp; the curve then becomes the common cardioid.

(2) Let $r = a \sin 3\theta$ be the equation to the curve.

$r = 0$ when $3\theta = n\pi$; that is for $\theta = 0$, $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$, $\theta = \pi$, $\theta = \frac{4\pi}{3}$, $\theta = \frac{5\pi}{3}$.

When $\theta = 2\pi$ or upwards the same series of values again recur. The curve therefore passes six times through the pole, and as r never becomes infinite, it must consist of six equal loops arranged symmetrically round that point. A little consideration will show that the form of the curve is that given in fig. 49.

This curve belongs to a class represented by the general equation $r = a \sin m\theta$, the properties of which have been very elaborately treated of by the Abbé Grandi, in a paper in the *Philosophical Transactions* for 1723, and in a book called rather quaintly *Flores Geometrici.* From a fanciful notion that these curves resembled the petals of roses, he gave them the name of "Rhodoneæ," and endeavoured to trace analogies between them and the flowers after which he had named them. The first paragraph of his paper in the *Philosophical Transactions* will give an idea of his way of treating the subject: "Suos Geometria hortos habet in quibus, æmula (an potius magistra?) naturæ, ludere solet, sua ipsius manu flores elegantissimos serens irrigans enutriens; quorum contemplatione cultores suos quandoque recreat ac summa voluptate perfundit."

(3) Let the curve be

$$r = a(\sin 2\theta - \sin\theta) = a\sin\theta\,(2\cos\theta - 1),$$

r is equal to 0 when $\sin\theta = 0$ and $\cos\theta = \frac{1}{2}$, or when

$$\theta = 0, \quad \theta = \pi, \quad \theta = \frac{\pi}{3}, \quad \theta = \frac{5\pi}{3}.$$

The values of r recur when $\theta = 2\pi$; and as r never becomes infinite, it appears that there are four loops arranged round the pole, one pair being smaller than the other.

From $\theta = 0$ to $\theta = \frac{1}{3}\pi$, r is positive.

From $\theta = \frac{1}{3}\pi$ to $\theta = \pi$, r is negative as $2\cos\theta - 1$ is negative, and $\sin\theta$ is positive.

From $\theta = \pi$ to $\theta = \frac{5\pi}{3}$, r is positive, since both factors are negative.

From $\theta = \frac{5\pi}{3}$ to $\theta = 2\pi$, r is negative.

The form of the curve is easily seen to be that in (fig. 50).

(4) Let the equation to the curve be

$$r = a(\tan\theta - 1) \qquad \text{(fig. 48)}.$$

When $\theta = 0$, $r = -a = OB$ if OA be measured in the positive direction.

From $\theta = 0$ to $\theta = \frac{1}{4}\pi$, r is negative and decreasing, and traces out the portion BDO of the curve.

When $\theta = \frac{1}{4}\pi$, $r = 0$, and the curve passes through the pole, cutting the axis at an angle of 45°.

From $\theta = \frac{1}{4}\pi$ to $\theta = \frac{1}{2}\pi$, r is positive and increasing, and traces out the portion OEL.

When $\theta = \frac{1}{2}\pi$, $r = \infty$. To see whether this corresponds to an asymptote we must find $r^2 \dfrac{d\theta}{dr}$.

Now $$\frac{dr}{d\theta} = a\,(1 + \tan^2\theta)\,;$$

therefore $$r^2 \frac{d\theta}{dr} = \frac{a^2(\tan\theta - 1)^2}{a\,(1 + \tan^2\theta)} = \frac{a\,(\sin\theta - \cos\theta)^2}{(\sin^2\theta + \cos^2\theta)} = a,$$

when $\theta = \frac{1}{2}\pi$. Therefore AL drawn perpendicular to the axis at a distance $OA = a$ is an asymptote to the curve.

From $\theta = \dfrac{\pi}{2}$ to $\theta = \dfrac{5\pi}{4}$, r is negative and diminishing, and it traces out the portion $KHACO$: the prolongation AK of AL being an asymptote to this branch.

When $\theta = \dfrac{5\pi}{4}$, $r = 0$, and the curve again passes through the pole, cutting the axis at an angle of 45°.

From $\theta = \dfrac{5\pi}{4}$ to $\theta = \dfrac{3\pi}{2}$, r is positive, and traces out the portion OFN; and when $\theta = \frac{1}{4}\pi$, $r = \infty$, and it is seen as before that a line BN perpendicular to the axis is an asymptote.

From $\theta = \dfrac{3\pi}{2}$ to $\theta = 2\pi$, r is negative and diminishing, and traces out the portion MGB.

When $\theta = 2\pi$ the curve joins on to the first portion, and is therefore complete. It is obviously unnecessary to consider negative values of θ as they are included in what has already been done.

(5) Let the equation to the curve be

$$r^2 = a^2 \frac{\sin 3\theta}{\cos\theta}.$$

The form of this curve is given in fig. 46.

CHAPTER XII.

ON THE CURVATURE OF CURVED LINES.

SECT. 1. *Radius of Curvature.*

WHEN the curve is referred to rectangular co-ordinates, if ρ be the radius of curvature

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}},$$

x being made the independent variable; and

$$\rho = \frac{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}},$$

y being made the independent variable; and

$$\frac{1}{\rho} = \left(\frac{d^2x}{ds^2} + \frac{d^2y}{ds^2}\right)^{\frac{1}{2}},$$

the arc being made the independent variable.

Ex. (1) In the parabola, the equation to which is

$$y^2 = 4mx,$$

$$\rho = \frac{2(m+x)^{\frac{3}{2}}}{m^{\frac{1}{2}}}.$$

Ex. (2) In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$$\rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}, \text{ where } e = \frac{(a^2 - b^2)^{\frac{1}{2}}}{a}.$$

(3) In the rectangular hyperbola referred to its asymptotes

$$xy = m^2,$$

and
$$\rho = \frac{(x^2 + y^2)^{\frac{3}{2}}}{-2m^2}.$$

(4) In all the curves of the second order the radius of curvature varies as the cube of the normal.

If N be the length of the normal, $N = y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$;

and therefore
$$\rho = \frac{N^3}{-y^3\dfrac{d^2y}{dx^2}},$$

All the curves of the second order are included in the equation

$$y^2 = 2px + qx^2;$$

therefore
$$y\frac{dy}{dx} = p + qx,$$

and
$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = q,$$

$$y^3\frac{d^2y}{dx^2} = qy^2 + y^2\left(\frac{dy}{dx}\right)^2 = -p^2.$$

Therefore
$$\rho = \frac{N^3}{p^2}.$$

(5) In the cubical parabola $3a^2y = x^3$,

$$\rho = -\frac{(a^4 + x^4)^{\frac{3}{2}}}{2a^4x}.$$

(6) In the semi-cubical parabola $3ay^2 = 2x^3$,

$$\rho = -\frac{(2a + 3x)^{\frac{3}{2}}x^{\frac{1}{2}}}{3^{\frac{1}{2}}.a}.$$

(7) In the cycloid $\dfrac{dy}{dx} = \dfrac{(2ay - y^2)^{\frac{1}{2}}}{y}$;

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{2a}{y}, \quad \frac{d^2y}{dx^2} = -\frac{a}{y^2}, \quad \rho = 2(2ay)^{\frac{1}{2}}.$$

(8) In the catenary $y = \frac{c}{2}(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}})$,

$$\frac{dy}{dx} = \frac{1}{2}(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}), \quad \frac{d^2y}{dx^2} = \frac{y}{c^2}, \quad \rho = -\frac{y^2}{2c}.$$

(9) In the tractory $y + (a^2 - y^2)^{\frac{1}{2}}\frac{dy}{dx} = 0$.

Taking the expression for ρ in which y is the independent variable we find,

$$\rho = \frac{a(a^2 - y^2)^{\frac{1}{2}}}{y}.$$

(10) In the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $\quad \rho = -3(axy)^{\frac{1}{3}}$.

If the curve be referred to polar co-ordinates r and θ, then

$$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}};$$

or, if it be expressed by the relation between r and the perpendicular on the tangent (p),

$$\rho = r\frac{dr}{dp}.$$

(11) In the cardioid $r = a(1 - \cos\theta)$,

$$\rho = \frac{(8ar)^{\frac{1}{2}}}{3}.$$

(12) In the lemniscate of Bernoulli $r^2 = a^2\cos 2\theta$,

$$\frac{dr}{d\theta} = -\frac{(a^4 - r^4)^{\frac{1}{2}}}{r}, \quad \left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^2} - r^2,$$

$$\frac{d^2r}{d\theta^2} = -\frac{a^4}{r^3} - r, \quad r\frac{d^2r}{d\theta^2} = -\frac{a^4 + r^4}{r^2},$$

$$\rho = \frac{a^2}{3r}.$$

(13) In the spiral of Archimedes $r = a\theta$,

$$\rho = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}.$$

(14) In the hyperbolic spiral $r = \frac{a}{\theta}$,

$$\rho = \frac{r(a^2 + r^2)^{\frac{3}{2}}}{a^3}.$$

(15) The equation to the lituus being $r^2 = \frac{a^2}{\theta}$,

$$\rho = \frac{r(4a^4 + r^4)^{\frac{3}{2}}}{2a^2(4a^4 - r^4)}.$$

(16) The equation to the trisectrix being $r = a(2\cos\theta \pm 1)$,

$$\rho = a\frac{(5 \pm 4\cos\theta)^{\frac{3}{2}}}{3(3 \pm 2\cos\theta)}.$$

(17) In the logarithmic spiral when referred to p and r,

$$p = mr, \qquad \rho = \frac{r}{m} = \frac{p}{m^2}.$$

(18) In the involute of the circle $p^2 = r^2 - a^2$, and $\rho = p$.

(19) The equation to Cotes' spirals is $p = \frac{br}{(a^2 + r^2)^{\frac{1}{2}}}$,

$$\rho = \frac{r(a^2 + r^2)^{\frac{3}{2}}}{a^2 b}.$$

(20) In the epicycloid

$$p^2 = \frac{c^2(r^2 - a^2)}{c^2 - a^2}.$$

Therefore $\rho = p\frac{c^2 - a^2}{c^2} = \frac{(c^2 - a^2)^{\frac{1}{2}}(r^2 - a^2)^{\frac{1}{2}}}{c}$.

SECT. 2. *Evolutes of Curves.*

When a curve is referred to rectangular co-ordinates, the co-ordinates (α, β) of its centre of curvature are given by the equations

$$\alpha = x - \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \frac{dy}{dx}, \quad \beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}.$$

To determine the equation to the evolute it is necessary to eliminate x and y between these equations and that of the given curve; but the complication of the formulæ renders this elimination always very troublesome, and most frequently impracticable. The few cases in which it can be effected we shall give.

(1) In the parabola $y^2 = 4ax$, whence

$$\alpha = 3x + 2a, \quad \beta = -\frac{y^3}{4a^2};$$

$$\text{therefore} \quad x = \frac{\alpha - 2a}{3}, \quad y = (-4a^2\beta)^{\frac{1}{3}}.$$

Substituting these values in the equation to the parabola, we find

$$(4a^2\beta)^{\frac{2}{3}} = \frac{4a}{3}(\alpha - 2a),$$

$$\text{or} \quad 27a\beta^2 = 4(\alpha - 2a)^3,$$

the equation to the semi-cubical parabola.

(2) In the rectangular hyperbola referred to its asymptotes

$$xy = m^2,$$

$$\text{whence} \quad 2\alpha = 3x + \frac{y^2}{x}, \quad 2\beta = 3y + \frac{x^2}{y}.$$

Adding

$$2(\alpha + \beta) = 3(x + y) + \frac{y^3 + x^3}{xy} = \frac{y^3 + 3x^2y + 3y^2x + x^3}{m^2},$$

$$\text{or} \quad 2m^2(\alpha + \beta) = (x + y)^3, \quad \text{or} \quad x + y = (2m^2)^{\frac{1}{3}}(\alpha + \beta)^{\frac{1}{3}}.$$

Similarly, subtracting

$$2m^2(\alpha - \beta) = (x - y)^3, \quad \text{or} \quad x - y = (2m^2)^{\frac{1}{3}}(\alpha - \beta)^{\frac{1}{3}}.$$

Adding and subtracting,

$$2x = (2m^2)^{\frac{1}{3}} \left\{(\alpha + \beta)^{\frac{1}{3}} + (\alpha - \beta)^{\frac{1}{3}}\right\},$$

$$2y = (2m^2)^{\frac{1}{3}} \left\{(\alpha + \beta)^{\frac{1}{3}} - (\alpha - \beta)^{\frac{1}{3}}\right\}.$$

Multiplying these together, and observing that $xy = m^2$, the equation to the evolute is found to be

$$(\alpha + \beta)^{\frac{2}{3}} - (\alpha - \beta)^{\frac{2}{3}} = (4m)^{\frac{2}{3}}.$$

(3) The equation to the evolute of the ellipse may be found in the same manner, but it is obtained more readily by considering it as the locus of the ultimate intersections of consecutive normals, as follows:

Let α, β, be the current co-ordinates of the normal, x, y of the curve; then if

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

be the equation to the ellipse, the equation to a normal passing through the point x, y is

$$\frac{a^2 \alpha}{x} - \frac{b^2 \beta}{y} = a^2 - b^2. \qquad (2)$$

Differentiating (2) and (1) with respect to x and y,

$$\frac{a^2 \alpha}{x^2}\, dx - \frac{b^2 \beta}{y^2}\, dy = 0, \qquad \frac{x\,dx}{a^2} + \frac{y\,dy}{b^2} = 0,$$

$$\text{whence} \quad \lambda \frac{x}{a^2} = \frac{a^2 \alpha}{x^2}, \quad \lambda \frac{y}{b^2} = -\frac{b^2 \beta}{y^2}.$$

Multiply by x, y and add, then

$$\lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{a^2 \alpha}{x} - \frac{b^2 \beta}{y}, \quad \text{or } \lambda = (a^2 - b^2).$$

$$\text{Therefore} \quad x^3 = \frac{a^4 \alpha}{a^2 - b^2}, \quad y^3 = -\frac{b^4 \beta}{a^2 - b^2},$$

and substituting the values of x and y in (1), we find

$$(a\,\alpha)^{\frac{2}{3}} + (b\,\beta)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}.$$

(4) The equation to the semi-cubical parabola is

$$3ay^2 = x^3,$$

whence $\alpha = -\left(x + \dfrac{3x^2}{2a}\right), \quad \beta = 4(a + x)\left(\dfrac{x}{3a}\right)^{\frac{1}{2}};$

and eliminating x, we obtain for the equation to the evolute

$$81a\beta^2 = 16\left\{2a \pm (a^2 - 6a\alpha)^{\frac{1}{2}}\right\}^2 \left\{\pm (a^2 - 6a\alpha)^{\frac{1}{2}} - a\right\}.$$

(5) In the hypocycloid, the equation to which is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

$$\alpha = x + 3x^{\frac{1}{3}}y^{\frac{2}{3}}, \quad \beta = y + 3x^{\frac{2}{3}}y^{\frac{1}{3}}.$$

Adding these

$$\alpha + \beta = x + y + 3(xy)^{\frac{1}{3}}(x^{\frac{1}{3}} + y^{\frac{1}{3}})$$

$$= (x^{\frac{1}{3}} + y^{\frac{1}{3}})\left\{x^{\frac{2}{3}} + 2(xy)^{\frac{1}{3}} + y^{\frac{2}{3}}\right\} = (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3.$$

Subtracting

$$\alpha - \beta = x - y - 3(xy)^{\frac{1}{3}}(x^{\frac{1}{3}} - y^{\frac{1}{3}}) = (x^{\frac{1}{3}} - y^{\frac{1}{3}})^3.$$

Whence

$$x^{\frac{1}{3}} + y^{\frac{1}{3}} = (\alpha + \beta)^{\frac{1}{3}}, \quad x^{\frac{1}{3}} - y^{\frac{1}{3}} = (\alpha - \beta)^{\frac{1}{3}}.$$

Adding and subtracting these equations

$$2x^{\frac{1}{3}} = (\alpha + \beta)^{\frac{1}{3}} + (\alpha - \beta)^{\frac{1}{3}},$$

$$2y^{\frac{1}{3}} = (\alpha + \beta)^{\frac{1}{3}} - (\alpha - \beta)^{\frac{1}{3}}.$$

Substituting in the equation to the hypocycloid

$$\left\{(\alpha + \beta)^{\frac{1}{3}} + (\alpha - \beta)^{\frac{1}{3}}\right\}^2 + \left\{(\alpha + \beta)^{\frac{1}{3}} - (\alpha - \beta)^{\frac{1}{3}}\right\}^2 = 4a^{\frac{2}{3}},$$

$$\text{or} \quad (\alpha + \beta)^{\frac{2}{3}} + (\alpha - \beta)^{\frac{2}{3}} = 2a^{\frac{2}{3}},$$

which is the equation to the evolute.

The following method sometimes allows us to find at least the differential equation of the evolute when direct elimination would be impracticable. If we can express $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of one of the variables only, we can sometimes express y or x in terms β or α: then without finding the other variable it is sufficient to substitute the values of y or x in the equation $\frac{d\beta}{d\alpha} = -\frac{dx}{dy}$, and we have a differential equation in α or β, which is the differential equation to the evolute.

(6) In the cycloid $\frac{dy}{dx} = \frac{(2ay - y^2)^{\frac{1}{2}}}{y}$,

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{2a}{y}, \quad \frac{d^2y}{dx^2} = -\frac{a}{y^2};$$

whence $y - \beta = 2y$, or $y = -\beta$.

Substituting this value in $\frac{d\beta}{d\alpha} = -\frac{dx}{dy}$, we have

$$\frac{d(-\beta)}{d\alpha} = \frac{(-\beta)}{\{2a(-\beta) - (-\beta)^2\}^{\frac{1}{2}}},$$

which is the equation to an equal and similar cycloid, but in an inverted position.

(7) The equation to the tractory is

$$\frac{dy}{dx} = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}},$$

$$\text{whence } \beta = \frac{a^2}{y}, \quad y = \frac{a^2}{\beta};$$

Substituting this value in $\frac{d\beta}{d\alpha} = -\frac{dx}{dy}$ we find

$$\frac{d\alpha}{d\beta} = \frac{a}{(\beta^2 - a^2)^{\frac{1}{2}}},$$

which is the differential equation to the catenary.

(8) The equation to the catenary is

$$\frac{dy}{dx} = \frac{c}{(2cx + x^2)^{\frac{1}{2}}};$$

whence $x - a = -(c + x)$ and $x = \frac{a - c}{2}$,

substituting in $\frac{d\beta}{da} = -\frac{dx}{dy}$,

$$2c\frac{d\beta}{da} + \{(a + c)^2 - 4c^2\}^{\frac{1}{2}} = 0,$$

which is therefore the differential equation to the evolute.

(9) The equation to the logarithmic curve is $y = a\epsilon^{\frac{x}{a}}$;

whence $y - \beta = -\frac{a^2 + y^2}{y}$, or $y^2 - \frac{\beta}{2}y + \frac{a^2}{2} = 0$.

From this $y = \frac{\beta \pm (\beta^2 - 8a^2)^{\frac{1}{2}}}{4}$;

$$\text{and} \quad 4a\frac{da}{d\beta} + \beta \pm (\beta^2 - 8a^2)^{\frac{1}{2}} = 0$$

is the equation to the evolute.

In curves referred to polar co-ordinates the most convenient mode of finding the equation to the evolute is by the relation between p and r.

If p and r be the co-ordinates of the curve,
$p_{,}$ and $r_{,}$ .... evolute,
ρ be the radius of curvature; then $p = f(r)$ being the equation to the curve,

$$r_{,}^2 = r^2 + \rho^2 - 2\rho p,$$

$$p_{,}^2 = r^2 - p^2, \quad \rho = r\frac{dr}{dp}.$$

Between these four equations we can eliminate ρ, r, p, and so find a relation between $p_{,}$ and $r_{,}$, which is the equation to the evolute.

(10) Let $p^2 = r^2 - a^2$.

$$\text{Then } \rho = p, \quad r_{,}^2 = r^2 + p^2 - 2p^2$$

$$= r^2 - p^2 = a^2,$$

$$\text{and } p_{,}^2 = r^2 - p^2 = a^2.$$

Hence $p_{,}$ and $r_{,}$ being both constants, the evolute is a circle.

(11) In the logarithmic spiral $p = mr$,

$$\text{whence} \quad \rho = \frac{r}{m}, \quad p_{,}^2 = r^2(1 - m^2),$$

$$r_{,}^2 = r^2 + \frac{r^2}{m^2} - 2r^2 = r^2\,\frac{(1 - m^2)}{m^2},$$

$$\text{and} \quad p_{,} = mr_{,},$$

the equation to a similar logarithmic spiral.

The logarithmic spiral may even be its own evolute; that is, one convolution of the curve may be the evolute of another convolution. To find the condition that this should be the case, let $r = \epsilon^{\frac{\theta}{a}}$ be the equation to the curve. Let P (fig. 52) be a point in the curve, PN the normal at that point touching a point Q in the convolution which is the evolute of the convolution AP. Then since the curve makes a constant angle with its radius vector, the angle SPT must be equal to the angle SQP; that is, PSQ must be a right angle. Hence the radius SQ is separated from the radius SP by some whole number of circumferences together with three right angles, or if

$$ASP = \theta, \quad ASQ = \theta - (4r + 3)\frac{\pi}{2}.$$

If $SP = r$, and $SQ = r_{,}$,

$$r = \epsilon^{\frac{\theta}{a}}, \quad r_{,} = \epsilon^{\frac{\theta}{a} - \frac{4r+3}{a}\frac{\pi}{2}}.$$

But Q being a point in the evolute, $r = ar_{,}$, so that

$$\epsilon^{\frac{\theta}{a}} = a\epsilon^{\frac{\theta}{a} - \frac{4r+3}{a}\frac{\pi}{2}};$$

$$\text{whence} \quad a = \epsilon^{\frac{4r+3}{a}\frac{\pi}{2}},$$

$$\text{or} \quad a^a = \epsilon^{(4r+3)\frac{\pi}{2}},$$

which is the condition that the parameter a must satisfy in order that the spiral whose equation is $r = \epsilon^{\frac{\theta}{a}}$ may be its own evolute.

(12) In the Epicycloid $p^2 = \dfrac{c^2 (r^2 - a^2)}{c^2 - a^2}$,

$$\rho = \frac{c^2 - a^2}{c^2} p, \quad p_,^2 = \frac{a^2}{c^2 - a^2} (c^2 - r^2),$$

$$r_,^2 = r^2 + \frac{(c^2 - a^2)^2}{c^4} p^2 - 2 \frac{c^2 - a^2}{c^2} p^2$$

$$= r^2 + \left(\frac{c^2 - a^2}{c^2} - 2\right) (r^2 - a^2)$$

$$= \frac{a^2}{c^2} (a^2 + c^2 - r^2).$$

Substituting for $c^2 - r^2$ its value in terms of $p_,$,

$$r_,^2 = \frac{a^4 + (c^2 - a^2)}{c^2} p_,^2,$$

$$\text{and} \quad p_,^2 = \frac{c^2 \left(r_,^2 - \dfrac{a^4}{c^2}\right)}{c^2 - a^2},$$

which is also the equation to an epicycloid.

CHAPTER XIII.

APPLICATIONS OF THE DIFFERENTIAL CALCULUS TO GEOMETRY OF THREE DIMENSIONS.

SECT. 1. *Tangencies.*

IF $$F(x, y, z) = 0$$
be the equation to a curved surface, the equation to the tangent plane at a point x, y, z is

$$(x' - x)\frac{dF}{dx} + (y' - y)\frac{dF}{dy} + (z' - z)\frac{dF}{dz} = 0,$$

where x', y', z' are the current co-ordinates of the tangent plane, x, y, z those of the point of contact.

If the equation to the surface consist of a function homogeneous of n dimensions in x, y, z equated to a constant, the equation to the tangent plane becomes

$$x'\frac{dF}{dx} + y'\frac{dF}{dy} + z'\frac{dF}{dz} = nc,$$

$F(x, y, z) = c$ being the equation to the surface.

If p be the perpendicular from the origin on the tangent plane,

$$p = \frac{x\dfrac{dF}{dx} + y\dfrac{dF}{dy} + z\dfrac{dF}{dz}}{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}};$$

and if the function be homogeneous of n dimensions,

$$p = \frac{nc}{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

The equations to a normal at a point x, y, z are

$$\frac{x'-x}{\dfrac{dF}{dx}} = \frac{y'-y}{\dfrac{dF}{dy}} = \frac{z'-z}{\dfrac{dF}{dz}}.$$

Ex. (1) The equation to the Ellipsoid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

that to the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

The perpendicular on the tangent plane from the origin is given by the equation

$$\frac{1}{p} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}}.$$

If we wish to find the locus of the intersection of the tangent plane with the perpendicular on it from the centre, we have to combine the equation to the tangent plane,

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

with the equations of a line perpendicular to it, and passing through the origin

$$\frac{a^2x'}{x} = \frac{b^2y'}{y} = \frac{c^2z'}{z}.$$

These last may be put under the form

$$\frac{ax'}{\dfrac{x}{a}} = \frac{by'}{\dfrac{y}{b}} = \frac{cz'}{\dfrac{z}{c}} = (a^2x'^2 + b^2y'^2 + c^2z'^2)^{\frac{1}{2}},$$

$$\text{since } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Multiplying each term of the equation to the tangent plane by the corresponding member in these last expressions, x, y, z are eliminated, and we have for the locus of the intersections

$$x'^2 + y'^2 + z'^2 = (a^2x'^2 + b^2y'^2 + c^2z'^2)^{\frac{1}{2}}.$$

This is the equation to the surface of elasticity in the wave Theory of Light.

(2) Let the equation to the surface be

$$xyz = m^3.$$

The equation to the tangent plane is

$$\frac{x'}{x} + \frac{y'}{y} + \frac{z'}{z} = 3.$$

The intercepts on the tangents are

$$x'_0 = 3x, \quad y'_0 = 3y, \quad z'_0 = 3z,$$

and the volume of the pyramid included between the tangent plane and the co-ordinate planes is $\dfrac{9xyz}{2} = \dfrac{9a^3}{2}$.

The volume of this pyramid is smaller than that of any other pyramid formed with the co-ordinate planes by a plane passing through the point x, y, z.

The length of the perpendicular from the origin is given by

$$\frac{1}{p} = \frac{1}{3}\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right)^{\frac{1}{2}}.$$

(3) The equation to the Cono-Cuneus of Wallis is

$$(a^2 - x^2)\, y^2 - c^2 z^2 = 0,$$

and the equation to the tangent plane is therefore

$$y^2 x x' - (a^2 - x^2)\, y y' + c^2 z z' = x^2 y^2.$$

(4) The equation to the *héliçoide gauche* is

$$x \cos\left(\frac{2\pi z}{h}\right) - y \sin\left(\frac{2\pi z}{h}\right) = 0;$$

and the equation to the tangent plane is

$$h\,(xy' - yx') + 2\pi\,(x^2 + y^2)\, z' = 2\pi z\,(x^2 + y^2);$$

and the perpendicular on it is

$$p = \frac{2\pi r z}{(h^2 + 4\pi^2 r^2)^{\frac{1}{2}}}; \text{ where } r^2 = x^2 + y^2.$$

(5) The equation of the *héliçoide développable* is

$$x\sin\left\{\frac{2\pi z}{h}-\frac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a}\right\}+y\cos\left\{\frac{2\pi z}{h}-\frac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a}\right\}=a.$$

The cosine of the angle which the tangent plane makes with the plane of xy is

$$\frac{\dfrac{dF}{dz}}{\left\{\left(\dfrac{dF}{dx}\right)^2+\left(\dfrac{dF}{dy}\right)^2+\left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

Let $\dfrac{2\pi z}{h}-\dfrac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a}=\theta$, then

$$\frac{dF}{dx}=\sin\theta-\frac{x(x\cos\theta-y\sin\theta)}{a(x^2+y^2-a^2)^{\frac{1}{2}}},$$

$$\frac{dF}{dy}=\cos\theta-\frac{y(x\cos\theta-y\sin\theta)}{a(x^2+y^2-a^2)^{\frac{1}{2}}},$$

$$\frac{dF}{dz}=\frac{2\pi}{h}(x\cos\theta-y\sin\theta).$$

Now $(x\cos\theta-y\sin\theta)^2=x^2\cos^2\theta+y^2\sin^2\theta-2xy\sin\theta\cos\theta$, and from the equation to the surface

$$2xy\sin\theta\cos\theta=a^2-x^2\sin^2\theta-y^2\cos^2\theta;\text{ therefore}$$

$$(x\cos\theta-y\sin\theta)^2=x^2+y^2-a^2.\text{ Hence}$$

$$\frac{dF}{dx}=\sin\theta-\frac{x}{a},\qquad\frac{dF}{dy}=\cos\theta-\frac{y}{a},$$

$$\frac{dF}{dz}=\frac{2\pi}{h}(x^2+y^2-a^2)^{\frac{1}{2}}.$$

From these expressions the cosine of the inclination of the tangent plane to the plane of xy is found to be

$$\frac{2\pi a}{(h^2+4\pi^2a^2)^{\frac{1}{2}}}.$$

The inclination is therefore constant, and equal to that of the helix, which is the directrix of the surface.

(6) Let the surface be Fresnel's surface of elasticity, the equation to which is

$$a^2x^2 + b^2y^2 + c^2z^2 = (x^2 + y^2 + z^2)^2.$$

The equation to the tangent plane is

$$(2r^2 - a^2)\,xx' + (2r^2 - b^2)\,yy' + (2r^2 - c^2)\,zz' = r^4,$$

where $r^2 = x^2 + y^2 + z^2$.

The perpendicular from the centre on the tangent plane is

$$\frac{(x^2 + y^2 + z^2)^2}{(a^4x^2 + b^4y^2 + c^4z^2)^{\frac{1}{2}}}.$$

When a curved line in space is given by the equations of two of its projections,

$$x = \phi(z), \qquad y = \psi(z),$$

the equations to a tangent at the point x, y, z are

$$x' - x = \frac{dx}{dz}(z' - z), \qquad y' - y = \frac{dy}{dz}(z' - z).$$

The direction cosines of the tangent are

$$\frac{dx}{ds}, \qquad \frac{dy}{ds}, \qquad \frac{dz}{ds}.$$

The equation to the normal plane is

$$(x' - x)\,dx + (y' - y)\,dy + (z' - z)\,dz = 0.$$

The equation to the osculating plane is

$$(x' - x)\,(dy\,d^2z - dz\,d^2y) + (y' - y)\,(dz\,d^2x - dx\,d^2z)$$
$$+ (z' - z)\,(dx\,d^2y - dy\,d^2x) = 0.$$

(7) Let the given curve be the helix, the equations to which are

$$x = a\cos\frac{z}{h}, \qquad y = a\sin\frac{z}{h}.$$

The equations to the tangent are

$$h\,(x' - x) + y\,(z' - z) = 0, \qquad h\,(y' - y) - x\,(z' - z) = 0.$$

If θ be the angle which the tangent makes with the plane of xy,

$$\tan\theta = \frac{h}{a}, \text{ and is therefore constant.}$$

The equation to the normal plane is

$$xy' - yx' + h(z' - z) = 0.$$

In finding the equation to the osculating plane we may for simplicity assume $d^2z = 0$, that is, make z the independent variable. This assumption readily gives us as the equation to the osculating plane,

$$h(xy' - yx') + a^2(z' - z) = 0.$$

In both of these equations if we make $x' = 0$, $y' = 0$, we find $z' = z$; that is, both planes cut the axis of z at the same point, which is the corresponding co-ordinate of the point in the curve.

(8) Let a curve of double curvature be formed by the intersection of two cylinders, the axes of which cut each other at right angles.

The equations to the curve are

$$x^2 + z^2 = a^2, \quad y^2 + z^2 = b^2,$$

the point of intersection of the axes of the cylinders being taken as origin, and the axes as the axes of x and y.

The equations to the tangent are

$$xx' + zz' = a^2, \quad yy' + zz' = b^2.$$

The equation to the normal plane is

$$\frac{x'}{x} + \frac{y'}{y} - \frac{z'}{z} = 1.$$

The equation to the osculating plane is, making z the independent variable, and therefore $d^2z = 0$,

$$b^2x^3x' - a^2y^3y' + (a^2 - b^2)\,z^3z' = a^2b^2(a^2 - b^2).$$

When a curved line in space is not given by the equations to its projections, but by the equations to any two surfaces,

$$F(x, y, z) = 0, \quad F_1(x, y, z) = 0,$$

we have

$$\frac{dF}{dx}dx + \frac{dF}{dy}dy + \frac{dF}{dz}dz = 0,$$

$$\frac{dF_1}{dx}dx + \frac{dF_1}{dy}dy + \frac{dF_1}{dz}dz = 0,$$

from which equations we can determine $\frac{dx}{dz}$, $\frac{dy}{dz}$ in terms of x, y, z: and these values are then to be substituted in the equations to the tangent, and to the normal and osculating planes.

(9) Let the curve be that formed by the intersection of a sphere and an ellipsoid. It is determined by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x^2 + y^2 + z^2 = r^2.$$

From these we find

$$\frac{dx}{dz} = \frac{a^2}{c^2}\frac{b^2 - c^2}{a^2 - b^2}\frac{z}{x}, \quad \frac{dy}{dz} = \frac{b^2}{c^2}\frac{c^2 - a^2}{a^2 - b^2}\frac{z}{y};$$

therefore the equations to the tangent are

$$\frac{x(x' - x)}{a^2(b^2 - c^2)} = \frac{y(y' - y)}{b^2(c^2 - a^2)} = \frac{z(z' - z)}{c^2(a^2 - b^2)}.$$

The equation to the normal plane is

$$a^2(b^2 - c^2)\frac{x' - x}{x} + b^2(c^2 - a^2)\frac{y' - y}{y} + c^2(a^2 - b^2)\frac{z' - z}{z} = 0.$$

This curve is the spherical ellipse; that is, it is a curve described on the surface of a sphere such that the sum of the arcs of great circles drawn from any point in the curve to two fixed points on the surface of the sphere is constant.

(10) Let the curve of double curvature be the equable spherical spiral. This is formed by the intersection of a sphere with a right cylinder the radius of whose base is one half of that of the sphere, and which passes through the centre of the sphere. The equations to the curve are therefore

$$x^2 + y^2 + z^2 = 4r^2, \quad y^2 + x^2 = 2rx,$$

the axis of z being taken parallel to the axis of the cylinder, and the axis of x passing through the centre of the base of the cylinder. The equations to the tangent are

$$y\,(y' - y) = (r - x)\,(x' - x), \quad z\,(z' - z) = r\,(x' - x).$$

SECT. 2. *Curvature.*

If a curved surface be given by an equation of the form

$$z = f\,(x, y),$$

and if $\dfrac{dz}{dx} = p, \quad \dfrac{dz}{dy} = q, \quad 1 + p^2 + q^2 = k^2,$

$$\frac{d^2 z}{dx^2} = r, \quad \frac{d^2 z}{dx\,dy} = s, \quad \frac{d^2 z}{dy^2} = t,$$

the greatest and least radii of curvature of the normal sections passing through a point x, y, z are given by the equation

$$\rho^2\,(rt - s^2) - \rho k \left\{(1 + q^2)r - 2pqs + (1 + p^2)t\right\} + k^4 = 0,$$

where ρ is the radius of curvature.

If the surface be given by an equation of the form

$$F\,(x, y, z) = 0,$$

and if we put

$$\frac{dF}{dx} = U, \quad \frac{dF}{dy} = V, \quad \frac{dF}{dz} = W, \quad \left(\frac{dF}{dx}\right)^2 + \left(\frac{dF}{dy}\right)^2 + \left(\frac{dF}{dz}\right)^2 = P^2,$$

$$\frac{d^2 F}{dx^2} = u, \quad \frac{d^2 F}{dy^2} = v, \quad \frac{d^2 F}{dz^2} = w,$$

$$\frac{d^2 F}{dy\,dz} = u', \quad \frac{d^2 F}{dx\,dz} = v', \quad \frac{d^2 F}{dx\,dy} = w',$$

the equation for determining the radii of maximum and minimum curvature is

$$U^2\left(v - \frac{P}{\rho}\right)\left(w - \frac{P}{\rho}\right) + V^2\left(w - \frac{P}{\rho}\right)\left(u - \frac{P}{\rho}\right) + W^2\left(v - \frac{P}{\rho}\right)\left(u - \frac{P}{\rho}\right)$$
$$- 2u'VW\left(u - \frac{P}{\rho}\right) - 2v'UW\left(v - \frac{P}{\rho}\right) - 2w'UV\left(w - \frac{P}{\rho}\right)$$
$$- U^2 u'^2 - V^2 v'^2 - W^2 w'^2 + 2VWv'w' + 2UWu'w' + 2UVu'v' = 0.$$

This equation is much longer than the preceding, but from its symmetry it is more useful in practice, and is very frequently much simplified by the vanishing of some of the quantities which it contains.*

(1) Let the given surface be the Ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Here $$U = \frac{2x}{a^2}, \quad V = \frac{2y}{b^2}, \quad W = \frac{2z}{c^2},$$

$$u = \frac{2}{a^2}, \quad v = \frac{2}{b^2}, \quad w = \frac{2}{c^2},$$

$$u' = 0, \quad v' = 0, \quad w' = 0.$$

Also $$P = 2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}} = \frac{2}{p},$$

where p is the perpendicular from the centre on the tangent plane. Hence the equation becomes

$$\frac{x^2}{a^2(p\rho - a^2)} + \frac{y^2}{b^2(p\rho - b^2)} + \frac{z^2}{c^2(p\rho - c^2)} = 0,$$

or $$\rho^2 - \left\{a^2 + b^2 + c^2 - (x^2 + y^2 + z^2)\right\}\frac{\rho}{p} + \frac{a^2b^2c^2}{p^4} = 0.$$

From the last term of this it appears that the product of the greatest and least radii of curvature of normal sections is constant for all points for which the perpendicular on the tangent plane is constant.

(2) Let the surface be the paraboloid, the equation to which is

$$\frac{y^2}{a} + \frac{z^2}{a'} = x.$$

In this case

$$U = -1, \quad V = \frac{2y}{a}, \quad W = \frac{2z}{a'},$$

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* For a demonstration of this equation see *Cambridge Mathematical Journal*, Vol. I. p. 137.

$$u = 0, \qquad v = \frac{2}{a}, \qquad w = \frac{2}{a'},$$

$$u' = 0, \qquad v' = 0, \qquad w' = 0, \qquad P = \frac{x}{p},$$

p having the same meaning as before. Then the equation for determining the radii of maximum and minimum curvature becomes

$$\rho^2 - (a + a' + 4x)\frac{x}{4p}\rho + \frac{aa'x^4}{4p^4} = 0.$$

(3) Let the equation to the surface be

$$xyz = m^3.$$

$$\text{Here} \quad U = yz, \qquad V = xz, \qquad W = xy,$$

$$u = 0, \qquad v = 0, \qquad w = 0,$$

$$u' = x, \qquad v' = y, \qquad w' = z,$$

Substituting these values in the general equation for the radii of curvature, it becomes

$$\rho^2 + 2(x^2 + y^2 + z^2)\frac{\rho}{p} + \frac{27m^6}{p^4} = 0.$$

(4) The equation to the *héliçoide gauche* is

$$x \cos nz - y \sin nz = 0.$$

$$U = \cos nz, \quad V = -\sin nz, \quad W = -n(x \sin nz + y \cos nz),$$

$$u = 0, \qquad v = 0, \qquad w = 0,$$

$$u' = -n \cos nz, \quad v' = -n \sin nz, \quad w' = 0.$$

The equation for determining ρ is reduced to

$$n^2\rho^2 - \left\{1 + n^2(x^2 + y^2)\right\}^2 = 0,$$

or the two radii of curvature are equal, but of opposite signs.

In a curve of double curvature the radius of absolute curvature is given by the formula

$$\frac{1}{\rho} = \left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2\right\}^{\frac{1}{2}},$$

s being the arc.

(5) Let the curve be the helix, the equations to which are

$$x = a\cos\frac{z}{h}, \quad y = a\sin\frac{z}{h}.$$

From these we find

$$\frac{dx}{ds} = -\frac{y}{(a^2+h^2)^{\frac{1}{2}}}, \quad \frac{dy}{ds} = \frac{x}{(a^2+h^2)^{\frac{1}{2}}}, \quad \frac{dz}{ds} = \frac{h}{(a^2+h^2)^{\frac{1}{2}}}.$$

Whence $\dfrac{d^2x}{ds^2} = -\dfrac{x}{a^2+h^2}, \quad \dfrac{d^2y}{ds^2} = -\dfrac{y}{a^2+h^2}, \quad \dfrac{d^2z}{ds^2} = 0,$

and $\rho = \dfrac{a^2+h^2}{a}$, and is therefore constant.

(6) Let the curve be the equable spherical spiral, the equations to which are

$$x^2 + y^2 + z^2 = 4r^2, \quad x^2 + y^2 = 2rx.$$

From these we find

$$\frac{d^2x}{ds^2} = \frac{4r(r-x) - x^2}{r(2r+x)^2}, \quad \frac{d^2y}{ds^2} = -\frac{y}{r}\,\frac{5r+x}{(2r+x)^2},$$

$$\frac{d^2z}{ds^2} = -\frac{z}{(2r+x)^2}.$$

Substituting these values in the expression for the radius of curvature, we find after certain reductions

$$\frac{1}{\rho} = \frac{(10r+3x)^{\frac{1}{2}}}{(2r+x)^{\frac{3}{2}}}.$$

The lines of curvature at any point of a surface are found by combining the equation to the surface with the equation

$$U(dVdz - dWdy) + V(dWdx - dUdz) + W(dUdy - dVdx) = 0,$$

U, V, W having the same meanings as before.

Between this equation and the equation to the surface and its differential we can eliminate each of the variables and its differential in succession, and thus obtain the differential equations to the projections of the lines of curvature on the coordinate planes.

(7) Let the surface be the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad (1)$$

The lines of curvature are determined by combining this with the equation

$$(b^2-c^2)\,x\,dy\,dz+(c^2-a^2)\,y\,dz\,dx+(a^2-b^2)\,z\,dx\,dy=0. \qquad (2)$$

To eliminate z and dz, multiply by $\frac{z}{c^2}$, and substitute the values

$$\frac{z^2}{c^2}=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}, \quad \frac{z\,dz}{c^2}=-\frac{x\,dx}{a^2}-\frac{y\,dy}{b^2},$$

when we obtain

$$\frac{b^2-c^2}{b^2}xy\left(\frac{dy}{dx}\right)^2+\left\{\frac{a^2-c^2}{a^2}x^2-\frac{b^2-c^2}{b^2}y^2-(a^2-b^2)\right\}\frac{dy}{dx}-\frac{a^2-c^2}{a^2}xy=0,$$

as the differential equation of the projection of the lines of curvature on the plane of xy.

Mr Leslie Ellis* has found a symmetrical integral of the equation representing the lines of curvature in an ellipsoid, which I shall introduce in this place, though it more properly belongs to another branch of our subject.

If in equation (2) we put

$$\frac{x^2}{a^2}=u, \quad \frac{y^2}{b^2}=v, \quad \frac{z^2}{c^2}=w,$$

we find, after changing the differentials and multiplying by $\frac{4(uvw)^{\frac{1}{2}}}{abc}$, that it becomes

$$(b^2-c^2)\,u\,dv\,dw+(c^2-a^2)\,v\,dw\,du+(a^2-b^2)\,w\,du\,dv=0, \qquad (3)$$

with the relation

$$u+v+w=1. \qquad (4)$$

Differentiating (3), and observing that

$b^2-c^2+c^2-a^2+a^2-b^2=0$, we get

$$(b^2-c^2)u\,d(dv\,du)+(c^2-a^2)v\,d(dw\,du)+(a^2-b^2)w\,d(du\,dv)=0. \qquad (5)$$

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* *Cambridge Mathematical Journal*, Vol. II. p. 133.

This is satisfied by

$$dvdu = \frac{1}{f}, \quad dwdu = \frac{1}{g}, \quad dudv = \frac{1}{h}, \qquad (6)$$

f, g, h being constants. But from (4) we have

$$du + dv + dw = 0; \text{ and from (6)}$$

$$du = fdudvdw, \quad dv = gdudvdw, \quad dw = hdudvdw.$$

Hence $$f + g + h = 0,$$

establishing a relation between f, g, h.

Now equation (6) implies two linear equations connecting u, v, w. Therefore a *particular solution* of (3) is two linear equations connecting the three variables, but the given equation (4) is linear, and therefore the solution in question is the one congruent to the problem. The other linear equation is found by eliminating the differentials from (3) by means of (6). The result is

$$(b^2 - c^2)\frac{u}{f} + (c^2 - a^2)\frac{v}{g} + (a^2 - b^2)\frac{w}{h} = 0;$$

or, putting for u, v, w their values,

$$(b^2 - c^2)\frac{x^2}{a^2 f} + (c^2 - a^2)\frac{y^2}{b^2 g} + (a^2 - b^2)\frac{z^2}{c^2 h} = 0.$$

This is evidently the equation to a cone of the second degree, having its vertex in the centre of the ellipsoid; and the lines of curvature are determined by the intersection of this cone with the ellipsoid.

(8) Let the surface be the paraboloid

$$\frac{z^2}{a} + \frac{y^2}{a'} - x = 0.$$

The general differential equation to the lines of curvature will be found by combining this with

$$(a' - a)\,dz\,dy + 2y\,dz\,dx - 2z\,dy\,dx = 0.$$

Multiplying by z and eliminating that variable and its differential, we obtain for the differential equation of the projections of the lines of curvature on the plane of xy,

$$\frac{a'-a}{a'}\,y\left(\frac{dy}{dx}\right)^2 + \left(2x - \frac{a'-a}{2}\right)\frac{dy}{dx} - y = 0.$$

The equation to the projection on yz is

$$\frac{yz}{a}\left(\frac{dz}{dy}\right)^2 + \left(\frac{a'-a}{4} - \frac{z^2}{a} + \frac{y^2}{a'}\right)\frac{dy}{dz} - \frac{yz}{a'} = 0.$$

(9) Let the equation to the surface be

$$xyz = m^3.$$

Then $\qquad U = yz, \quad V = xz, \quad W = xy.$

Substituting these values in the general equation to lines of curvature, we find after some reductions,

$$x\,(y^2 - z^2)\,dz\,dy + y\,(z^2 - x^2)\,dx\,dz + z\,(x^2 - y^2)\,dx\,dy = 0,$$

which combined with the equation to the surface gives the lines of curvature.

(10) Dupin in his *Développements de Géométrie*, p. 322, has demonstrated the following very remarkable theorem relative to the lines of curvature on surfaces: "If there be three systems of surfaces which intersect each other at right angles, any two of them will trace on the third its lines of curvature."

Let the three systems of surfaces be represented by the equations

$$f(x,y,z) = c, \;(1)\quad f_1(x,y,z) = c_1, \;(2)\quad f_2(x,y,z) = c_2, \;(3)$$

c, c_1, c_2 being the variable parameters by which each individual in each system is distinguished.

If we represent the differentials of these equations taken with respect to x, y, z by U, V, W, the conditions for the surfaces intersecting at right angles are

$$UU_1 + VV_1 + WW_1 = 0, \qquad (4)$$

$$U_1U_2 + V_1V_2 + W_1W_2 = 0, \qquad (5)$$

$$U_2U + V_2V + W_2W = 0. \qquad (6)$$

Eliminating U_2, V_2, W_2 in turn between (5) and (6) we get

$$U_2 = k(VW_1 - V_1W),$$

$$V_2 = k(WU_1 - W_1U),$$

$$W_2 = k(UV_1 - U_1V),$$

where k is an unknown multiplier. Hence, as

$$U_2dx + V_2dy + W_2dz$$

is a perfect differential function, it follows that

$$(VW_1 - V_1W)\,dx + (WU_1 - W_1U)\,dy + (UV_1 - U_1V)\,dz,$$

may be integrated by means of a factor; and this is all the information which the equations (5) and (6) give with respect to the intersection of the surfaces (1) and (2).

By the ordinary condition of integrability by a factor, we have

$$\begin{aligned}(VW_1-V_1W)\left\{U_1\frac{dW}{dz}+W\frac{dU_1}{dz}-U\frac{dW_1}{dz}-W_1\frac{dU}{dz}-U\frac{dV_1}{dy}-V_1\frac{dU}{dy}+V\frac{dU_1}{dy}+U_1\frac{dV}{dy}\right\}+\\(WU_1-W_1U)\left\{V_1\frac{dU}{dx}+U\frac{dV_1}{dx}-V\frac{dU_1}{dx}-U\frac{dV_1}{dx}-W_1\frac{dV}{dz}-V\frac{dW_1}{dz}+W\frac{dV_1}{dz}+V_1\frac{dW}{dz}\right\}+\\(UV_1-U_1V)\left\{W_1\frac{dV}{dy}+V\frac{dW_1}{dy}-W\frac{dV_1}{dy}-V_1\frac{dW}{dy}-U_1\frac{dW}{dx}-W\frac{dU_1}{dx}+U\frac{dW_1}{dx}+W_1\frac{dU}{dx}\right\}=0.\end{aligned}$$

The coefficient of $\dfrac{dU}{dx}$ in this equation is $-U_1(VW_1-V_1W)$,

and that of $\dfrac{dU_1}{dx}$ .... is $U(VW_1-V_1W)$.

But since $Udx + Vdy + Wdz$ is a complete differential function we have

$$\frac{dU}{dy}=\frac{dV}{dx},\quad \frac{dU}{dz}=\frac{dW}{dx},\quad \frac{dV}{dz}=\frac{dW}{dy};$$

and similarly for U_1, V_1, W_1; therefore the equation may be put under the form

$$\begin{aligned}(VW_1-V_1W)\left\{U\frac{dU_1}{dx}+V\frac{dV_1}{dx}+W\frac{dW_1}{dx}-U_1\frac{dU}{dx}-V_1\frac{dV}{dx}-W_1\frac{dW}{dx}\right\}+\\(WU_1-W_1U)\left\{U\frac{dU_1}{dy}+V\frac{dV_1}{dy}+W\frac{dW_1}{dy}-U_1\frac{dU}{dy}-V_1\frac{dV}{dy}-W_1\frac{dW}{dy}\right\}+\\(UV_1-U_1V)\left\{U\frac{dU_1}{dz}+V\frac{dV_1}{dz}+W\frac{dW_1}{dz}-U_1\frac{dU}{dz}-V_1\frac{dV}{dz}-W_1\frac{dW}{dz}\right\}=0.\quad(7)\end{aligned}$$

Again, as (4) is identically true we may differentiate it with respect to x, y, z separately, when we have

$$U\frac{dU_1}{dx} + V\frac{dV_1}{dx} + W\frac{dW_1}{dx} = -U_1\frac{dU}{dx} - V_1\frac{dV}{dx} - W_1\frac{dW}{dx};$$

and similarly for the others. Hence equation (7) becomes

$$\begin{aligned}&(VW_1 - V_1W)\left\{U_1\frac{dU}{dx} + V_1\frac{dV}{dx} + W_1\frac{dW}{dx}\right\} + \\ &(WU_1 - W_1U)\left\{U_1\frac{dU}{dy} + V_1\frac{dV}{dy} + W_1\frac{dW}{dy}\right\} + \\ &(UV_1 - U_1V)\left\{U_1\frac{dU}{dz} + V_1\frac{dV}{dz} + W_1\frac{dW}{dz}\right\} = 0. \qquad (8)\end{aligned}$$

The curve which is the intersection of any surface of (1) with any surface of (2) satisfies the equations

$$U\,dx + V\,dy + W\,dz = 0, \qquad (9)$$

$$U_1dx + V_1dy + W_1dz = 0. \qquad (10)$$

By combining these we have

$$VW_1 - V_1W = \lambda dx, \quad WU_1 - UW_1 = \lambda dy, \quad UV_1 - U_1V = \lambda dz;$$

λ being an unknown quantity: and by combining (10) with (4) we have also

$$U_1 = \mu(Vdz - Wdy), \quad V_1 = \mu(Wdx - Udz), \quad W_1 = \mu(Udy - Vdx),$$

μ being an unknown quantity. Hence equation (8) becomes

$$U_1dU + V_1dV + W_1dW = 0, \text{ or}$$

$$(Vdz - Wdy)\,dU + (Wdx - Udz)\,dV + (Udy - Vdx)\,dW = 0.$$

But this is the equation to the lines of curvature on the surface (1), and from the symmetry of the equations (4), (5), (6) it is clear that a similar result may be obtained for the surfaces (2) and (3). Hence the three systems of surfaces intersect each other in their lines of curvature*.

In Liouville's *Journal de Mathématiques*, Vol. v. p. 313, the reader will find a memoir on Curvilinear Co-ordinates by

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* This demonstration of Dupin's Theorem was communicated to me by Mr Leslie Ellis.

Lamé, in which are demonstrated many very curious Theorems respecting the curvature of orthotomic surfaces.

SECT. 3. *Singular points and lines in Surfaces.*

Let $$F(x, y, z) = 0, \qquad (1)$$

be the equation to a surface; then the direction cosines of the tangent plane at a point x, y, z are

$$\frac{\dfrac{dF}{dx}}{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}, \quad \frac{\dfrac{dF}{dy}}{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}},$$

$$\frac{\dfrac{dF}{dz}}{\left\{\left(\dfrac{dF}{dx}\right)^2 + \left(\dfrac{dF}{dy}\right)^2 + \left(\dfrac{dF}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

If now we can find values of x, y, z which, satisfying equation (1), also make at the same time

$$\frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0, \qquad (2)$$

the position of the tangent plane at the point in question will become indeterminate, since the direction cosines then take the form $\frac{0}{0}$. At such a singular point we shall then have generally not a single tangent plane but many, even an infinite number, in which case their ultimate intersections will form a tangent cone, the vertex of which will be the singular point in question. If the three equations (2) are satisfied by assigning certain relations between the variables, then the curve formed by the intersection of the surface (1) with that indicated by the relation between the variables which satisfies equations (2) is a locus of singular points, that is to say, it is a line in which two or more sheets of the surface intersect.

If for possible values of two of the variables on one side of the singular point we find impossible values of the third variable, that point is a *cusp*. If the same occur at every point of the singular line, it is called an *edge of regression* (*arète de rebroussement*). Such for examples are the curves which are the loci of the ultimate intersections of the generating lines of developable surfaces.

To determine the equation to the tangent cone (if there be one) at a singular point, or the angle made by the tangent planes at the same point of a singular line, we proceed as follows. Giving the same designations as before to U, V, W, u, v, w, u', v', w', we have, by differentiating equation (1),

$$Udx + Vdy + Wdz = 0.$$

Differentiating again, we have

$$Ud^2x + Vd^2y + Wd^2z + udx^2 + vdy^2 + wdz^2$$
$$+ 2u'dydz + 2v'dxdz + 2w'dxdy = 0.$$

Now at a singular point, $U = 0$, $V = 0$, $W = 0$, and this equation is reduced to

$$(u)dx^2 + (v)dy^2 + (w)dz^2 + 2(u')dydz$$
$$+ 2(v')dxdz + 2(w')dxdy = 0, \qquad (3)$$

the bracketed letters indicating the values they take when we substitute for x, y, z their values at the point in question.

If all the quantities u, v, w, u', v', w' vanish, we must proceed to another differentiation, but in the examples which we shall adduce this will not be necessary. Now the equation (3) gives a relation subsisting between the increments dx, dy, dz in the surface at the singular point. These are the same for the surface and for a straight line touching it at the point; and therefore equation (3) gives a relation between the increments dx, dy, dz on the tangent lines at the singular points, or since this relation is the same for all points of these lines, we may substitute x, y, z for dx, dy, dz in (3), and we find

$$(u)x^2 + (v)y^2 + (w)z^2 + 2(u')yz + 2(v')xz + 2(w')xy = 0, \qquad (4)$$

as the equation to the locus of the tangent lines at the singular point which is taken as the origin of co-ordinates.

This equation, unless for particular values of the coefficients, is that to a cone of the second degree. If we had proceeded to the third differentials we should have found the equation to a cone of the third degree, and so on in succession. It may happen that the equation (4) may be decomposed into two factors of the first degree, and then it will represent two planes. The condition that this may be the case is

$$(u)(v)(w) - (u)(v')^2 - (v)(w')^2 - (w)(u')^2 + 2(u')(v')(w') = 0.$$

Ex. (1) Find the nature of the point at the origin in the surface

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 - c^2z^2.$$

Here $U = 2x(2r^2 - a^2)$,

$$V = 2y(2r^2 - b^2),$$

$$W = 2z(2r^2 + c^2),$$

$$u = 2(2r^2 - a^2) + 8x^2,$$

$$v = 2(2r^2 - b^2) + 8y^2,$$

$$w = 2(2r^2 + c^2) + 8z^2,$$

$$u' = 8yz, \quad v' = 8xz, \quad w' = 8xy.$$

Now when $x = 0$, $y = 0$, $z = 0$, U, V, W all vanish, while $u = -2a^2$, $v = -2b^2$, $w = 2c^2$, $u' = 0$, $v' = 0$, $w' = 0$, so that the equation to the tangent cone at the origin is

$$a^2x^2 + b^2y^2 - c^2z^2 = 0.$$

(2) The equation to Fresnel's wave-surface in biaxal crystals is

$$(x^2 + y^2 + z^2)(a^2x^2 + b^2y^2 + c^2z^2) - a^2(b^2 + c^2)x^2 - b^2(a^2 + c^2)y^2$$
$$- c^2(a^2 + b^2)z^2 + a^2b^2c^2 = 0;$$

find whether it has any singular points, and their nature.

Here $U = 2x\{a^2(r^2 - b^2 - c^2) + a^2x^2 + b^2y^2 + c^2z^2\}$,

$$V = 2y\{b^2(r^2 - a^2 - c^2) + a^2x^2 + b^2y^2 + c^2z^2\},$$

$$W = 2z\{c^2(r^2 - a^2 - b^2) + a^2x^2 + b^2y^2 + c^2z^2\},$$

where $r^2 = x^2 + y^2 + z^2$.

Now if we assume $y = 0$, $r^2 = b^2$ which involve

$$x = \pm c \left(\frac{a^2 - b^2}{a^2 - c^2}\right)^{\frac{1}{2}}, \quad z = \pm a \left(\frac{b^2 - c^2}{a^2 - c^2}\right)^{\frac{1}{2}};$$

these values will satisfy the equation to the surface, and will also make U, V, and W vanish: hence, as the double signs of x and z may be combined in four different ways, there are four singular points on the surface. To obtain the equation to the tangent cone we must find the values of u, v, w, u', v', w', at the singular points. These are readily seen to be

$$u = 8a^2c^2 \frac{a^2 - b^2}{a^2 - c^2}, \quad v = -2(a^2 - b^2)(b^2 - c^2), \quad w = 8a^2c^2 \frac{b^2 - c^2}{a^2 - c^2},$$

$$u' = 0, \quad v' = 4ac \left\{(a^2 - b^2)(b^2 - c^2)\right\}^{\frac{1}{2}} \frac{a^2 + c^2}{a^2 - c^2}, \quad w' = 0.$$

Substituting these values, and dividing the whole by

$$8a^2c^2 \frac{(a^2 - b^2)(b^2 - c^2)}{a^2 - c^2},$$

we find as the equation to the cone

$$\frac{x^2}{b^2 - c^2} - \frac{a^2 - c^2}{4a^2c^2} y^2 + \frac{z^2}{a^2 - b^2} + \frac{a^2 + c^2}{\left\{(a^2 - b^2)(b^2 - c^2)\right\}^{\frac{1}{2}}} \frac{xz}{ac} = 0.$$

The existence of these singular points in the wave-surface was first pointed out by Sir W. Hamilton.

(3) Let the equation to the surface be

$$z(x^2 + y^2 + z^2) + ax^2 + by^2 = 0,$$

$$U = 2x(z + a), \quad V = 2y(z + b), \quad W = x^2 + y^2 + 3z^2.$$

At the origin where $x = 0$, $y = 0$, $z = 0$, these three quantities vanish, therefore there is a singular point at the origin: also

$$u = 2(z + a), \quad v = 2(z + b), \quad w = 6z,$$

$$u' = 2y, \quad v' = 2x, \quad w' = 0,$$

$$(u) = 2a, \quad (v) = 2b, \quad (w) = 0, \quad (u') = 0, \quad (v') = 0, \quad (w') = 0.$$

The equation to the locus of the tangent lines becomes then

$$ax^2 + by^2 = 0,$$

which, a and b being supposed to be both positive, can only represent the axis of z. The cone in this case degenerates into a straight line; and as z can never be positive, since that renders x and y impossible, it appears that the point under consideration is a cusp. The surface surrounds the negative axis of z, which it touches at the origin, so that its form resembles the shape of the flower of the convolvulus.

If a and b be of contrary signs the equation to the locus of the tangent lines is

$$ax^2 - by^2 = 0,$$

which represents two planes perpendicular to the plane of x, y.

(4) Let the surface be the cono-cuneus of Wallis, the equation to which is

$$a^2y^2 - x^2(c^2 - z^2) = 0.$$

Here $U = -2x(c^2 - z^2)$, $V = 2a^2y$, $W = 2x^2z$.

These all vanish when $x = 0$, $y = 0$ independently of the value of z; hence the axis of z is a locus of singular points or a singular line.

$$u = -2(c^2 - z^2), \quad v = 2a^2, \quad w = 2x^2,$$

$$u' = 0, \quad v' = 4xz, \quad w' = 0.$$

The equation to the tangent lines becomes in this case

$$a^2y'^2 - (c^2 - z^2)x'^2 = 0,$$

where x', y' are accentuated to distinguish them from z, the undetermined co-ordinate of the point of contact. The preceding equation is equivalent to those of two planes perpendicular to the plane of xy,

$$ay' + (c^2 - z^2)^{\frac{1}{2}}x' = 0,$$

$$ay' - (c^2 - z^2)^{\frac{1}{2}}x' = 0,$$

By assigning different values to z we obtain different equations corresponding to successive points taken along the axis of z.

(5) The equation to the *héliçoide développable* is

$$x \sin\left\{\frac{2\pi z}{h} - \frac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a}\right\} + y\cos\left\{\frac{2\pi z}{h} - \frac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a}\right\} = a.$$

Putting $\dfrac{2\pi z}{h} - \dfrac{(x^2+y^2-a^2)^{\frac{1}{2}}}{a} = \theta$, we find as in a previous example

$$U = \sin\theta - \frac{x\,(x\cos\theta - y\sin\theta)}{a\,(x^2+y^2-a^2)^{\frac{1}{2}}};$$

$$V = \cos\theta - \frac{y\,(x\cos\theta - y\sin\theta)}{a\,(x^2+y^2-a^2)^{\frac{1}{2}}};$$

$$W = \frac{2\pi}{h}\,(x\cos\theta - y\sin\theta).$$

But we found before that

$$x\cos\theta - y\sin\theta = (x^2+y^2-a^2)^{\frac{1}{2}};$$

therefore if we assume

$$x = a\sin\frac{2\pi z}{h}, \quad y = a\cos\frac{2\pi z}{h},$$

the preceding expressions will vanish, and therefore the line determined by these equations, and the equation to the surface is a *locus* of singular points.

This line is the intersection of the surface by the cylinder

$$x^2 + y^2 = a^2,$$

and is evidently the generating helix. Since in the equation to the surface x^2+y^2 can never be less than a^2, it appears that no part of the surface lies within the helix, which is therefore truly an edge of regression.

On proceeding to the second differential coefficients, and substituting in them the critical values of x and y we find, retaining only the terms which become infinite from involving $(x^2+y^2-a^2)^{\frac{1}{2}}$ in the denominator,

$$(u) = -2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (v) = 2\sin\frac{2\pi z}{h}\cos\frac{2\pi z}{h}, \quad (w) = 0,$$

$$(u') = \frac{2\pi}{h}\,a\cos\frac{2\pi z}{h}, \quad (v') = \frac{2\pi}{h}\,a\sin\frac{2\pi z}{h}, \quad (w') = \sin^2\frac{2\pi z}{h} - \cos^2\frac{2\pi z}{h};$$

so that the equation to the locus of the tangent lines is

$$(y'^2 - x'^2)\, xy + x'y'(x^2 - y^2) + 2\pi \frac{a^2}{h} z' (x'x + y'y) = 0,$$

where the accentuated letters are the current co-ordinates of the tangents, and the unaccentuated the undetermined co-ordinates of the point of contact. This equation may be decomposed into two factors,

$$y'x - x'y + 2\pi \frac{a^2}{h} z' = 0,$$

$$x'x + y'y = 0,$$

which are the equations to two planes.

Umbilici. These are points at which the two principal radii of curvature are equal. The conditions for determining them are

$$\frac{1 + p^2}{r} = \frac{pq}{s} = \frac{1 + q^2}{t}.$$

(6) In the ellipsoid

$$p = -\frac{c^2 x}{a^2 z}, \quad q = -\frac{c^2 y}{b^2 z},$$

$$r = -\frac{c^4 (b^2 - y^2)}{a^2 b^2 z^3}, \quad s = -\frac{c^4 xy}{a^2 b^2 z^3}, \quad t = -\frac{c^4 (a^2 - x^2)}{a^2 b^2 z^3}.$$

Substituting these values in the conditions for an umbilicus, we have

$$\frac{b^2}{a^2} z \frac{a^4 z^2 + c^4 x^2}{c^4 (b^2 - y^2)} = z = \frac{a^2}{b^2} z \frac{b^4 z^2 + c^4 y^2}{c^4 (a^2 - x^2)};$$

these are satisfied by

$$y = 0, \quad x = \pm a \left(\frac{a^2 - b^2}{a^2 - c^2}\right)^{\frac{1}{2}}, \quad z = \pm c \left(\frac{b^2 - c^2}{a^2 - c^2}\right)^{\frac{1}{2}},$$

which are therefore the co-ordinates of four umbilici.

(7) Let the surface be the paraboloid

$$z = \frac{x^2}{a} + \frac{y^2}{a'},$$

$$p = \frac{2x}{a'}, \quad q = \frac{2y}{a'}, \quad r = \frac{2}{a}, \quad s = 0, \quad t = \frac{2}{a'}.$$

Hence we have

$$\frac{a^2+4x^2}{2a} = \frac{\dfrac{4xy}{aa'}}{0} = \frac{a'^2+4y^2}{2a'}.$$

In order that these equations may hold we must have either $x=0$, or $y=0$. Taking the former we find

$$\frac{2y^2}{a'} = \frac{a}{2} - \frac{a'}{2}, \quad \text{or } y = \tfrac{1}{2}\{a'(a-a')\}^{\frac{1}{2}}, \quad \text{and } z = \frac{a-a'}{4}.$$

Now if $a > a'$ the value of y is possible, and there are two umbilici, the co-ordinates of which are

$$x=0, \quad y = \pm\tfrac{1}{2}\{a'(a-a')\}^{\frac{1}{2}}, \quad z = \frac{a-a'}{4}.$$

If $a < a'$ we must take $y=0$, and then we find

$$x = \pm\tfrac{1}{2}\{a(a'-a)\}^{\frac{1}{2}}, \quad z = \frac{a'-a}{4}.$$

(8) In the surface, the equation to which is

$$xyz = m^3,$$

there is an umbilicus at the point

$$x=m, \quad y=m, \quad z=m.$$

CHAPTER XIV.

ENVELOPS TO LINES AND SURFACES.

THE earliest questions the solutions of which involved the Theory of Envelops or Ultimate Intersections were those which related to evolutes of curves, investigated by Huyghens,* and those relating to Caustics, a subject introduced by Tschirnhausen;† but these authors did not follow any general analytical method for the solution of such problems. Leibnitz was the first who considered the general theory of questions of this kind, so well adapted for exemplifying the utility of his Calculus; and in two memoirs in the *Acta Eruditorum*,‡ he gave a general process for the solution of all problems which depended on the successive intersections of lines whether straight or curved, the position or magnitude of which were changed according to some law. This method is the same as that usually employed, no important modification having been subsequently introduced, and may be stated in the following manner.

If $u = f(x, y, z, a, b, c \dots) = 0$ be the equation to a surface, $a, b, c \dots$ being parameters determining its position and magnitude, the envelop of all the surfaces formed by the variation of $a, b, c \dots$ is found by eliminating these quantities between the equations

$$u = 0, \quad \frac{du}{da} = 0, \quad \frac{du}{db} = 0, \quad \frac{du}{dc} = 0 \dots$$

When, as is often the case, there are one or more equations of condition between the parameters, the method of indeterminate multipliers may frequently be conveniently employed. The same method of course applies to lines in two dimensions.

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* *Opera*, Vol. I. p. 89.

† *Acta Eruditorum*, 1682.

‡ 1692, p. 168, and 1694, p. 311.

Ex. (1) Find the equation to the curve which touches all the straight lines determined by the equation

$$y = ax + \frac{m}{a},$$

where a is supposed to vary.

Here $$u = y - ax - \frac{m}{a} = 0,$$

$$\frac{du}{da} = -x + \frac{m}{a^2} = 0,$$

whence $a^2 = \frac{m}{x}$, and substituting this value we have

$$y = 2(mx)^{\frac{1}{2}}, \quad \text{or} \quad y^2 = 4mx,$$

the equation to a parabola.

(2) Find the equation to the curve which touches all the lines determined by the equation

$$y = ax + r(1 + a^2)^{\frac{1}{2}},$$

when a is supposed to vary.

Here $$u = y - ax - r(1 + a^2)^{\frac{1}{2}},$$

$$\frac{du}{da} = -\left\{x + \frac{ra}{(1 + a^2)^{\frac{1}{2}}}\right\} = 0.$$

Multiply by a and add to the original equation.

Then $$y = r\left\{(1 + a^2)^{\frac{1}{2}} - \frac{a^2}{(1 + a^2)^{\frac{1}{2}}}\right\} = \frac{r}{(1 + a^2)^{\frac{1}{2}}};$$

therefore $$y^2 = \frac{r^2}{1 + a^2}; \quad \text{also} \quad x^2 = \frac{r^2 a^2}{1 + a^2}.$$

Adding, we have $x^2 + y^2 = r^2$; the equation to a circle.

(3) Find the envelop of the series of parabolas whose equation is

$$y^2 = a(x - a),$$

a being the variable parameter.

Here $\frac{du}{da} = 0$ gives $2a - x = 0$, or $a = \frac{x}{2}$;

$$\text{whence} \quad y^2 = \frac{x^2}{4}, \quad \text{or } y = \pm \frac{x}{2},$$

the equations to two straight lines.

(4) To find the envelop of the series of ellipses defined by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{(k-a)^2} = 1.$$

$$\text{Here} \quad \frac{du}{da} = 0 \text{ gives } \frac{x^2}{a^3} - \frac{y^2}{(k-a)^3} = 0;$$

$$\text{whence} \quad a = \frac{kx^{\frac{2}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}, \quad k - a = \frac{ky^{\frac{2}{3}}}{x^{\frac{2}{3}} - y^{\frac{2}{3}}};$$

and on substituting these values in the original equation we find as the equation to the envelop

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}.$$

(5) The straight line PQ (fig. 41) slides between the rectangular axes Ax, Ay; find the locus of its ultimate intersections.

Let $AP = a$, $AQ = b$, $PQ = c$; then the equation to PQ is

$$\frac{x}{a} + \frac{y}{b} = 1,$$

a, b being subject to the condition

$$a^2 + b^2 = c^2.$$

Differentiating with respect to a and b,

$$\frac{xda}{a^2} + \frac{ydb}{b^2} = 0 \quad (1), \qquad ada + bdb = 0 \quad (2),$$

λ (1) − (2) = 0 gives on equating to zero the coefficients of each differential.

$$\lambda \frac{x}{a^2} = a, \quad \lambda \frac{y}{b^2} = b.$$

Multiply by a, b, respectively, and add; then by the first two equations,

$$\lambda \left(\frac{x}{a} + \frac{y}{b}\right) = \lambda = a^2 + b^2 = c^2;$$

therefore $a^3 = c^2 x$, $b^3 = c^2 y$,

and, substituting these values of a and b in the equation of condition, we obtain

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

as the locus of the ultimate intersections of PQ.

(6) If the equation to a straight line be

$$\frac{x}{a} + \frac{y}{b} = 1,$$

a and b being subject to the condition

$$\frac{a}{m} + \frac{b}{n} = 1,$$

the locus of its ultimate intersections is

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}},$$

which is the equation to a parabola referred to two tangents as axes.

(7) Find the envelop to the series of parabolas determined by the equation

$$y = ax - (1 + a^2)\frac{x^2}{4c},$$

where a is the variable parameter.

The result is a parabola, the equation to which is

$$x^2 = 4c(c - y).$$

This is the equation to the curve touched by the parabolas described by projectiles discharged from a given point with a constant velocity, but at different inclinations to the horizon. The problem was proposed by Fatio to John Bernoulli, who solved it, but not by any general method: it was the first case which was brought forward of the locus of the ultimate intersections of *curved* lines.—*Commercium Epistolicum of Leibnitz and Bernoulli*, Vol. I. p. 17.

(8) Find the curve which is constantly touched by the circles determined by the equation

$$(x - a)^2 + y^2 = b^2,$$

a and b being the co-ordinates of a parabola, so that

$$b^2 = 4ma.$$

The resulting equation is

$$y^2 = 4m(x + m),$$

which is the equation to an equal parabola, the vertex of which is shifted through a distance $-m$.

(9) Find the envelop of the series of ellipses defined by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{a^2}{m^2} + \frac{b^2}{n^2} = 1.$$

The resulting equation is

$$\pm \frac{x}{m} \pm \frac{y}{n} = 1.$$

The equations to four straight lines in the space contained by which all the ellipses lie.

(10) Find the locus of the ultimate intersections of chords joining the extremities of conjugate diameters of an ellipse the axes of which are a and b.

If x', y' be the co-ordinates of the extremity of a diameter, $\frac{a}{b}y'$, $-\frac{b}{a}x'$ are the co-ordinates of the extremity of its conjugate. Hence the equation to the line joining their extremities is

$$x'\left(y - \frac{b}{a}x\right) - y'\left(x + \frac{a}{b}\right)y + ab = 0,$$

x' and y' being connected by the equation to the ellipse

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1.$$

The resulting equation of the locus of the ultimate intersections is

$$\frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 1;$$

the equation to an ellipse, the axes of which are $\frac{a}{2^{\frac{1}{2}}}$, $\frac{b}{2^{\frac{1}{2}}}$.

(11) If from every point in a curve of the second order pairs of tangents be drawn to another curve of the second order, find the curve which is constantly touched by the chord of contact.

Let us suppose for simplicity that the second curve is an ellipse referred to its centre, its equation being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{I.}$$

Let the co-ordinates of a point from which a pair of tangents to (I) is drawn be α, β, then the equation to the chord of contact is

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} = 1: \qquad \text{II.}$$

α, β are supposed to be the co-ordinates of a point which is always in a curve of the second order: they are therefore connected by the equation

$$A\alpha^2 + 2B\alpha\beta + C\beta^2 + 2D\alpha + 2E\beta + 1 = 0, \qquad \text{III.}$$

Now to find the curve which is constantly touched by (II) differentiate (II) and (III) with respect to α and β.

$$\frac{x d\alpha}{a^2} + \frac{y d\beta}{b^2} = 1, \qquad (1)$$

$$(A\alpha + B\beta + D)\, d\alpha + (B\alpha + C\beta + E) = 0: \qquad (2)$$

$\lambda\,(1) + (2) = 0$ gives us

$$\lambda \frac{x}{a^2} + A\alpha + B\beta + D = 0,$$

$$\lambda \frac{y}{b^2} + B\alpha + C\beta + E = 0.$$

Multiply by α, β and add, then by (II) and (III)

$$\lambda = D\alpha + E\beta + 1.$$

Substituting in the preceding equations we have

$$(D\alpha + E\beta + 1)\frac{x}{a^2} + A\alpha + B\beta + D = 0, \qquad (3)$$

$$(D\alpha + E\beta + 1)\frac{y}{b^2} + B\alpha + C\beta + E = 0. \qquad (4)$$

Between (II), (3), and (4) we can eliminate α, β, and we obtain the final equation

$$(C - E^2)\frac{x^2}{a^4} - 2(B - DE)\frac{xy}{a^2b^2} + (A - D^2)\frac{y^2}{b^4},$$

$$+ 2(CD - BE)\frac{x}{a^2} + 2(AE - BD)\frac{y}{b^2} + AC - B^2 = 0. \quad \text{IV.}$$

This being of the second order, it appears that the locus of the ultimate intersections of (I) is a conic section. This is a case of the general problem of *reciprocal polars.* The curve (I) is called the directrix, the point α, β its pole; and the line (II) the polar with reference to $(\alpha, \beta.)$ The curves (III), (IV) are the reciprocal polars, and possess a great number of corresponding properties of considerable interest, but the nature of this work precludes us from entering on that subject. The reader who is curious in such matters will find memoirs on these related curves by Poncelet, in the *Annales de Gergonne*, Vol. VIII. p. 201, and Bobillier, *Ib.* Vol. XIX. p. 106, and p. 302. He will also find these questions along with others of a similar kind very ingeniously treated, in a short tract on "Tangential Co-ordinates," by *J. Booth of Trinity College, Dublin.* The method employed by that author does not come within the scope of the present work, but it merits attention, as affording a ready solution of many curious problems which yield with difficulty to the power of ordinary analysis.

(12) A plane whose equation is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,$$

a, b, c being subject to the condition

$$abc = m^3,$$

will always touch the surface whose equation is

$$xyz = \frac{m^3}{27}.$$

(13) To find the envelop of the system of spheres determined by the equations

$$(x - a)^2 + (y - b)^2 + z^2 = r^2, \qquad a^2 + b^2 = c^2.$$

Differentiating with respect to a and b,

$$(x - a)\,da + (y - b)\,db = 0 \quad (1), \qquad a\,da + b\,db = 0\ (2);$$

$\lambda(2) + (1) = 0$ gives on equating to zero the coefficients of each differential,

$$\lambda a + (x - a) = 0 \quad (3), \qquad \lambda b + (y - b) = 0 \quad (4).$$

$$\text{whence} \quad ay = bx, \quad \text{or} \quad \frac{x}{a} = \frac{y}{b},$$

$a(3) + b(4)$ gives

$$\lambda c^2 + ax + by - c^2 = 0.$$

But as

$$\frac{x}{a} = \frac{y}{b},$$

$$\frac{ax + by}{a^2 + b^2} = \frac{ax + by}{c^2} = \pm\frac{(x^2 + y^2)^{\frac{1}{2}}}{c},$$

$$\text{whence} \quad \lambda = \frac{c \pm (x^2 + y^2)^{\frac{1}{2}}}{c}.$$

Substituting this value of λ in (3) and (4), squaring and adding,

$$\{c \pm (x^2 + y^2)^{\frac{1}{2}}\}^2 = (x - a)^2 + (y - b)^2 = r^2 - z^2,$$

by the original equation; and this is the equation to the envelop.

(14) To find the surface always touched by a plane which cuts off from a right cone an oblique cone of constant volume.

Taking the vertex of the cone as origin, and its axis as the axis of z, the equation to the cone is

$$x^2 + y^2 = c^2 z^2 \qquad (1)$$

where c is the tangent of the half angle of the cone.

The equation to the cutting plane is

$$lx + my + nz = v, \qquad (2)$$

l, m, n being the cosines of the angles which it makes with the co-ordinate planes, so that

$$l^2 + m^2 + n^2 = 1, \qquad (3)$$

and v being the perpendicular from the origin on the plane.

Extracting the square root of (1) and substituting in it the value of z from (2), we have

$$(x^2 + y^2)^{\frac{1}{2}} = \frac{cv}{n} - \frac{c\,(lx + my)}{n},$$

which is the equation to the projection on (xy) of the section of (1) by (2); and as the radius vector is a rational function of x and y, the origin, that is, the vertex of the cone, must be the focus of the projection. Comparing it with the general equation to a conic section referred to, its focus

$$r = \frac{a\,(1 - e^2)}{1 + e\cos(\theta - a)},$$

$$\text{or}\ \ (x^2 + y^2)^{\frac{1}{2}} = a\,(1 - e^2) - e\cos a\,x - e\sin a\,y,$$

we find

$$a\,(1 - e^2) = \frac{cv}{n},\quad e^2 = \frac{c^2\,(l^2 + m^2)}{n^2};$$

whence the area of the projection is

$$\frac{\pi n c^2 v^2}{\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{3}{2}}},$$

and the area of the section is therefore

$$\frac{\pi\, c^2\, v^2}{\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{3}{2}}}.$$

The volume of the oblique cone cut off is

$$\frac{\pi c^2}{3}\,\frac{v^3}{\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{3}{2}}},$$

which is to be constant. Neglecting the constant multiplier and extracting the cube root, we may put

$$\frac{v}{\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{1}{2}}} = a,\ \text{ or }\ v = a\,\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{1}{2}},\qquad (4)$$

We therefore have the equation

$$lx + my + nz = a\,\{n^2 - c^2\,(l^2 + m^2)\}^{\frac{1}{2}},$$

l, m, n being connected by the equation

$$l^2 + m^2 + n^2 = 1.$$

The result of the elimination of l, m, n is

$$x^2 + y^2 - c^2z^2 = ac\left\{c^2z^2 - (x^2 + y^2)\right\}^{\frac{1}{2}},$$

or $$\left\{c^2z^2 - (x^2 + y^2)\right\}^{\frac{1}{2}}\left[ac + \left\{c^2z^2 - (x^2 + y^2)\right\}^{\frac{1}{2}}\right] = 0.$$

The factor $$ac + \left\{c^2z^2 - (x^2 + y^2)\right\}^{\frac{1}{2}} = 0,$$

is the equation to the required envelop.

Transposing and squaring, this becomes

$$c^2z^2 - (x^2 + y^2) = a^2c^2,$$

the equation to a hyperboloid of revolution of two sheets, the possible axis of which coincides with the axis of z.

If the theory of reciprocal polars given in Ex. 11, be applied to the surfaces of the second order, it will be found that the reciprocal polar of a surface of the second order is also a surface of the second order; and that when the one surface can be generated by the motion of a straight line, the other can be so generated also. For the properties of reciprocal polars in surfaces the reader may consult the memoirs indicated in Ex. 11, and also one by Brianchon, *Journal de l'Ecole Polytechnique*, Vol. VI. p. 308.

(15) Find the surface traced out by the ultimate intersections of the planes which touch the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

along the curve made by its intersection with the plane

$$lx + my + nz = \delta.$$

If x', y', z' be the current co-ordinates of the tangent plane, its equation is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

where x, y, z are supposed to vary subject to the previous conditions. Differentiating we have

$$\frac{xdx}{a^2} + \frac{ydy}{b^2} + \frac{zdz}{c^2} = 0, \qquad (1)$$

$$ldx + mdy + ndz = 0, \qquad (2)$$

$$\frac{x'dx}{a^2} + \frac{y'dy}{b^2} + \frac{z'dz}{c^2} = 0 : \qquad (3)$$

$\lambda(1) + \mu(3) + (2) = 0$ gives, on equating to zero the coefficients of each differential,

$$\lambda\frac{x}{a^2} + \mu\frac{x'}{a^2} + l = 0, \quad \lambda\frac{y}{b^2} + \mu\frac{y'}{b^2} + m = 0,$$

$$\frac{\lambda z}{c^2} + \mu\frac{z'}{c^2} + n = 0.$$

Multiply by x, y, z, and add, then by the equations of condition, $\lambda + \mu + \delta = 0$.

Substituting for λ in the preceding equations they become $\mu(x - x') = a^2l - \delta x$, $\mu(y - y') = b^2m - \delta y$, $\mu(z - z') = c^2n - \delta z$, whence

$$\frac{x - x'}{a^2l - \delta x} = \frac{y - y'}{b^2m - \delta y} = \frac{z - z'}{c^2n - \delta z} = \rho, \text{ suppose.}$$

Now multiplying numerator and denominator of these fractions by $\frac{x'}{a^2}$, $\frac{y'}{b^2}$, $\frac{z'}{c^2}$ respectively, and adding together the numerators and the denominators,

$$\rho = \frac{1 - \left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2}\right)}{lx' + my' + nz' - \delta}.$$

But on multiplying the numerator and denominator of these fractions by l, m, n respectively, and adding the numerators and the denominators, we also have

$$\rho = \frac{\delta - (lx' + my' + nz')}{a^2l^2 + b^2m^2 + c^2n^2 - \delta^2}.$$

Therefore equating the two values of ρ we have

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} + \frac{z'^2}{c^2} - 1 = \frac{\{\delta - (lx' + my' + nz')\}^2}{a^2l^2 + b^2m^2 + c^2n^2 - \delta^2},$$

as the required equation to the surface.

(16) Find the equation to the surface which is constantly touched by the plane

$$lx + my + nz = v,$$

l, m, n, v being connected by the equations

$$l^2 + m^2 + n^2 = 1,$$

$$\frac{l^2}{v^2 - a^2} + \frac{m^2}{v^2 - b^2} + \frac{n^2}{v^2 - c^2} = 0.$$

Differentiating with respect to l, m, n, v we have,

$$(1) \quad x\,dl + y\,dm + z\,dn = dv.$$

$$(2) \quad l\,dl + m\,dm + n\,dn = 0.$$

$$(3) \quad \frac{l\,dl}{v^2 - a^2} + \frac{m\,dm}{v^2 - b^2} + \frac{n\,dn}{v^2 - c^2}$$

$$= v\,dv \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\},$$

$\lambda\,(1) = \mu\,(2) + (3)$ gives, on equating the coefficients of each differential,

$$(4) \quad \lambda x = \mu l + \frac{l}{v^2 - a^2}.$$

$$(5) \quad \lambda y = \mu m + \frac{m}{v^2 - b^2}.$$

$$(6) \quad \lambda z = \mu n + \frac{n}{v^2 - c^2}.$$

$$(7) \quad \lambda = v \left\{ \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2} \right\};$$

$l\,(4) + m\,(5) + n\,(6)$ gives by the conditions,

$$(8) \quad \lambda v = \mu,$$

$x\,(4) + y\,(5) + z\,(6)$ gives

$$\lambda r^2 = \mu v + \frac{lx}{v^2 - a^2} + \frac{my}{v^2 - b^2} + \frac{nz}{v^2 - c^2};$$

whence $(9) \quad \lambda\,(r^2 - v^2) = \dfrac{lx}{v^2 - a^2} + \dfrac{my}{v^2 - b^2} + \dfrac{nz}{v^2 - c^2};$

$(4)^2 + (5)^2 + (6)^2$ gives

$$\lambda^2 r^2 = \mu^2 + \frac{l^2}{(v^2 - a^2)^2} + \frac{m^2}{(v^2 - b^2)^2} + \frac{n^2}{(v^2 - c^2)^2};$$

whence (10) $\lambda^2(r^2 - v^2) = \frac{\lambda}{v}$ by (7) and (8);

and therefore $\lambda = \frac{1}{v(r^2 - v^2)}$, and $\mu = \frac{1}{r^2 - v^2}$.

Substituting these values in (4) we have

$$\frac{x}{v(r^2 - v^2)} = l\left(\frac{1}{r^2 - v^2} + \frac{1}{v^2 - a^2}\right);$$

whence $$\frac{x}{r^2 - a^2} = \frac{vl}{v^2 - a^2}.$$

Similarly $$\frac{y}{r^2 - b^2} = \frac{vm}{v^2 - b^2};$$

and $$\frac{z}{r^2 - c^2} = \frac{vn}{v^2 - c^2}.$$

Multiply by x, y, z and add, then by (9) and (10)

$$\frac{x^2}{r^2 - a^2} + \frac{y^2}{r^2 - b^2} + \frac{z^2}{r^2 - c^2} = 1.$$

This is the equation to the surface of a wave of light propagated through a crystalline medium. See Fresnel, *Mémoires de l' Institut*, Vol. VII. p. 136; Ampère *Annales de Chimie et de Physique*, Vol. XXXIX. p. 113; and Smith, *Cambridge Transactions*, Vol. VI. p. 85.

If from the above equation we subtract

$$\frac{x^2 + y^2 + z^2}{r^2} = 1,$$

and reduce, we find

$$\frac{a^2x^2}{r^2 - a^2} + \frac{b^2y^2}{r^2 - b^2} + \frac{c^2z^2}{r^2 - c^2} = 0,$$

which is the form of the equation given by Fresnel.

CHAPTER XV.

GENERAL THEOREMS IN THE DIFFERENTIAL CALCULUS.

In this chapter I shall collect those Theorems in the Differential Calculus which, depending only on the laws of combination of the symbols of differentiation, and not on the functions which are operated on by these symbols, may be proved by the method of the separation of the symbols: but as the principles of this method have not as yet found a place in the elementary works on the Calculus, I shall first state briefly the theory on which it is founded.

There are a number of theorems in ordinary algebra, which, though apparently proved to be true only for symbols representing numbers, admit of a much more extended application. Such theorems depend only on the laws of combination to which the symbols are subject, and are therefore true for all symbols, whatever their nature may be, which are subject to the same laws of combination. The laws with which we have here concern are few in number, and may be stated in the following manner. Let a, b represent two operations, u, v two subjects on which they operate, then the laws are

$$(1) \qquad ab(u) = ba(u),$$

$$(2) \qquad a(u+v) = a(u) + a(v),$$

$$(3) \qquad a^m . a^n . u = a^{m+n} . u.$$

The first of these laws is called the *commutative law*, and symbols which are subject to it are called commutative symbols. The second law is called *distributive*, and the symbols subject to it distributive symbols. The third law is not so much a law of combination of the operation denoted by a, but rather of the operation performed on a, which is indicated by the index affixed to a. It may be conveniently called the law of repetition, since the most obvious and important case of it is that in which m and n are integers, and a^m therefore indicates the repetition m times of the operation a. That these are the laws employed in the demonstration of the principal theorems in Algebra, a slight examination of the processes will easily shew; but they are not confined to symbols of numbers; they apply also to the symbol used to denote differentiation. For if u be a function of two variables x and y, we have by known theorems in the Differential Calculus,

$$\frac{d}{dx}\cdot\frac{d}{dy}(u) = \frac{d}{dy}\cdot\frac{d}{dx}(u);$$

Also considering u and v as functions of x only,

$$\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v),$$

and besides

$$\left(\frac{d}{dx}\right)^m \cdot \left(\frac{d}{dx}\right)^n = \left(\frac{d}{dx}\right)^{m+n}(u).$$

The principal theorems in Algebra which depend on these laws, and which have therefore analogues in the Differential Calculus, are the Binomial Theorem with the great number of theorems—Exponential, Logarithmic, and others, which are derived from it; and the theorem of the decomposition of a multinomial of any order into simple factors with the various consequences which are deduced from it.

It is to be observed that in all the applications of this method to the Differential Calculus, a constant has the same laws of combination with the differentials that they have with each other, and therefore the theorems are true for complex symbols involving constants and symbols of differentiation. Also, there are two ways in which symbols of differentiation may differ from each other, either by having reference to different variables in the same function, or by having reference to different functions of the same variable, and this difference gives rise to two totally distinct series of theorems as will be seen in the following examples.

It is worthy of remark, that the indices in the greater number of these theorems may be any whatever: I shall not however make any use of the interpretation of the formulæ when the indices of differentiation are fractional. It is easy to see that when they are negative they are equivalent to integrals of a corresponding positive degree: for by the law of indices,

$$\left(\frac{d}{dx}\right)^{-n} \cdot \left(\frac{d}{dx}\right)^{m} u = \left(\frac{d}{dx}\right)^{m-n} u.$$

Also $$\int^n dx^n \left(\frac{d}{dx}\right)^{m} u = \left(\frac{d}{dx}\right)^{m-n} u,$$

and therefore

$$\left(\frac{d}{dx}\right)^{-n} = \int^n dx^n;$$

this interpretation I shall frequently have occasion to use.

The principle of the method of the separation of symbols of operation from their subjects was first correctly given by Servois, in the *Annales des Mathématiques*, Vol. v. p. 93. Some very valuable researches on this subject by Mr Murphy will be found in the *Philosophical Transactions* for 1837.

(1) *Taylor's Theorem.* This theorem may be reduced into a very convenient shape by the separation of the symbols: for as

$$f(x+h) = f(x) + \frac{h}{1}\frac{d}{dx}.f(x) + \frac{h^2}{1.2}\left(\frac{d}{dx}\right)^2 f(x) + \frac{h^3}{1.2.3}\left(\frac{d}{dx}\right)^3 f(x) + \&c.,$$

we have, by placing the function outside,

$$f(x+h) = \left\{1 + \frac{h}{1}\frac{d}{dx} + \frac{h^2}{1.2}\left(\frac{d}{dx}\right)^2 + \frac{h^3}{1.2.3}\left(\frac{d}{dx}\right)^3 + \&c.\right\} f(x).$$

Now it is easily seen that the series of operations on the second side of the equation follows the law of the expansion of the exponential ϵ^{hx} in terms of hx, and as the symbol $\frac{d}{dx}$ is subject to the same laws of combination as the symbol x is supposed to be subject to in the demonstration of the exponential Theorem, we may consistently write the preceding equation under the form

$$f(x+h) = \epsilon^{h\frac{d}{dx}} f(x).$$

As we shall have frequent occasion to speak of this operation of converting $f(x)$ into $f(x+h)$ it will be convenient to denote it by a single symbol, and that which, following M. Servois, we shall employ is E, but as it is necessary to distinguish the value of the increment we must attach to the symbol E the letter h. We might write therefore

$$f(x+h) = E_h f(x) = \epsilon^{h\frac{d}{dx}} f(x).$$

Farther consideration, however, shews us that the symbol h is subject to the index law and may therefore be written as indices usually are. For as

$$E_h . f(x) = f(x+h),$$

if k be another increment

$$E_k E_h f(x) = E_k f(x+h) = f(x+h+k) = E_{h+k} f(x),$$

which is the index law. We may, therefore, put

$$f(x+h) = E^h . f(x),$$

and throughout our operations consider h as an index.

(2) *Binomial Theorem* for differentials with respect to different variables.

If u be a function of two variables x and y, we have

$$d(u) = \frac{du}{dx} dx + \frac{du}{dy} dy;$$

or, separating the symbol of operation from the subject,

$$d(u) = \left(\frac{d}{dx} dx + \frac{d}{dy} dy\right) u.$$

Affixing the general symbol n as an index to the operations on both sides of the equation, we have

$$d^n(u) = \left(\frac{d}{dx} dx + \frac{d}{dy} dy\right)^n u.$$

Expanding the operation on the second side by the Binomial Theorem, since the demonstration of that Theorem supposes only that the symbols are subject to the laws of combination before laid down, there results

$$d^n(u) = \frac{d^n u}{dx^n} dx^n + n\frac{d^n u}{dx^{n-1}\,dy} dx^{n-1}\,dy$$

$$+ n\frac{(n-1)}{1\,.\,2}\,\frac{d^n u}{dx^{n-2}dy^2}\,dx^{n-2}\,dy^2 + \&\text{c.}$$

(3) In the same way, by means of the Multinomial Theorem, we may shew that if u be a function of any number of variables x, y, z...

$$d^n(u) = 1\,.\,2\,...\,n\Sigma\frac{d^n u}{dx^\alpha\,dy^\beta dz^\gamma...}\,\frac{dx^\alpha\,dy^\beta\,dz^\gamma...}{1.2...\alpha.1.2...\beta.1.2...\gamma...},$$

where $\alpha + \beta + \gamma + \&\text{c.} = n$.

(4) By the Theory of equations it is shewn that the expression

$$x^n + A_1\,x^{n-1} + A_2\,x^{n-2} + \&\text{c.} + A_{n-1}\,x + A_n$$

is equivalent to

$$(x - a_1)\,(x - a_2)\,...\,(x - a_n)\,;$$

a_1, a_2, ... a_n being the roots of the expression equated to zero. It follows therefore that

$$\frac{d^n u}{dx^n} + A_1\frac{d^n u}{dx^{n-1}\,dy} + A_2\frac{d^n u}{dx^{n-2}\,dy^2} + \&\text{c.} + A_n\frac{d^n u}{dy^n}$$

is equal to

$$\left(\frac{d}{dx} - a_1\frac{d}{dy}\right)\left(\frac{d}{dx} - a_2\frac{d}{dy}\right)...\left(\frac{d}{dx} - a_n\frac{d}{dy}\right)u,$$

a_1, a_2 ... a_n having the same meanings as before.

In this Theorem it is necessary that none of the quantities A_1 ... A_n should contain u, x or y.

(5) If u be a function of one variable x only, the preceding Theorem becomes

$$\frac{d^n u}{dx^n} + A_1\frac{d^{n-1}u}{dx^{n-1}} + A_2\frac{d^{n-2}u}{dx^{n-2}} + \&\text{c.} + Au$$

$$= \left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\left(\frac{d}{dx} - a_3\right)......\left(\frac{d}{dx} - a_n\right)u.$$

(6) By the theory of the decomposition of rational fractions, we know that

$$\{x^n + A_1 x^{n-1} + \text{\&c.} + A_n\}^{-1} = \frac{1}{x^n + A_1 x^{n-1} + A_2 x^{n-2} \dots + A_n}$$

$$= \frac{N_1}{x-a_1} + \frac{N_2}{x-a_2} + \frac{N_3}{x-a_3} + \dots\dots + \frac{N_n}{x-a_n},$$

when a_1, a_2, $a_3 \dots a_n$ are the roots (supposed all unequal) of

$$x^n + A_1 x^{n-1} + \text{\&c.} + A_n = 0,$$

and $$N_1 = \frac{1}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)},$$

with similar expressions for N_2, N_3, &c. ... N_n.

It follows therefore that

$$\left\{\left(\frac{d}{dx}\right)^n + A_1\left(\frac{d}{dx}\right)^{n-1} + A_2\left(\frac{d}{dx}\right)^{n-2} + \text{\&c.} + A_n\right\}^{-1} u$$

$$= N_1\left(\frac{d}{dx}-a_1\right)^{-1} u + N_2\left(\frac{d}{dx}-a_2\right)^{-1} u + \dots + N_n\left(\frac{d}{dx}-a_n\right)^{-1} u.$$

Or if u be a function of two variables, x and y,

$$\left\{\left(\frac{d}{dx}\right)^n + A_1\frac{d^n}{dx^{n-1}dy} + A_2\frac{d^n}{dx^{n-2}dy^2} + \dots + A_n\left(\frac{d}{dy}\right)^n\right\}^{-1} u$$

$$= N_1\left(\frac{d}{dy}\right)^{-(n-1)}\left(\frac{d}{dx}-a_1\frac{d}{dy}\right)^{-1} u + N_2\left(\frac{d}{dy}\right)^{-(n-1)}\left(\frac{d}{dx}-a_2\frac{d}{dy}\right)^{-1} u + \dots$$

$$+ N_n\left(\frac{d}{dy}\right)^{-(n-1)}\left(\frac{d}{dx}-a_n\frac{d}{dy}\right)^{-1} u.$$

If we suppose r of the quantities a to be equal to each other, they will give rise to a series of p terms of the form $M_p\left(\frac{d}{dx}-a\frac{d}{dy}\right)^{-p} u$ where p receives all integer values from 1 to r. The value of the coefficient M_p is easily found. For if we put

$$f(z) = (z-a)^r \phi(z),$$

$$M_p = \frac{1}{1.2\dots(r-p)}\left(\frac{d}{dz}\right)^{r-p} \phi(z) \text{ when } z = a.$$

The results contained in the preceding four Examples are of great use in the Integration of Linear Differential Equations, and in the sequel I shall have frequent occasion to employ them. The theorem in Ex. 6 was first given by Mr George Boole of Lincoln, in the *Cambridge Mathematical Journal*, Vol. II. p. 114.

(7) *Binomial Theorem* for differentials with respect to different functions.

If u and v be two functions of x, then

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

Now if we accentuate the symbol of differentiation which applies to v to distinguish it from that which applies to u, we may write

$$\frac{d}{dx}(uv) = \left(\frac{d}{dx} + \frac{d'}{dx}\right)uv.$$

Affixing the index n to the symbols of operation on both sides,

$$\left(\frac{d}{dx}\right)^n (uv) = \left(\frac{d}{dx} + \frac{d'}{dx}\right)^n uv;$$

or expanding the binomial on the second side by the theorem of Newton, we have

$$\left(\frac{d}{dx}\right)^n (uv) = v\frac{d^n u}{dx^n} + n\frac{dv}{dx}\frac{d^{n-1}u}{dx^{n-1}} + n\frac{(n-1)}{1.2}\frac{d^2v}{dx^2}\frac{d^{n-2}u}{dx^{n-2}} + \&c.$$

This is the theorem of Leibnitz who arrived at it by induction for integer indices; but it is true whether n be integer or fractional, positive or negative.

(8) This theorem may be extended to the product of any number of functions by means of the multinomial theorem, so that we have

$$\left(\frac{d}{dx}\right)^n (uvw\ldots) = 1.2\ldots n\Sigma.\left\{\frac{\left(\frac{d}{dx}\right)^\alpha u.\left(\frac{d}{dx}\right)^\beta v.\left(\frac{d}{dx}\right)^\gamma w\ldots}{1.2\ldots\alpha.1.2\ldots\beta.1.2\ldots\gamma}\right\},$$

where $\alpha + \beta + \gamma + \ldots = n$.

(9) If n be negative in the theorem of Leibnitz, $\left(\frac{d}{dx}\right)^{-n}(uv) = \int^n dx^n (uv)$, and therefore

$$\int^n dx^n(uv) = v\int^n dx^n u - n\frac{dv}{dx}\int^{n-1} dx^{n-1} u + \frac{n(n+1)}{1.2}\frac{d^2v}{dx^2}\int^{n-2} dx^{n-2} u - \&c.$$

which is the general formula for integration by parts.

(10) In the last expression let $u = 1$; then

$$\int^n dx^n u = \frac{x^n}{1.2\ldots n};$$

and

$$\int^n dx^n(v) = \frac{x^{n-1}}{1.2\ldots n-1}\left(\frac{x}{n}v - \frac{x^2}{n+1}\frac{dv}{dx} + \frac{1}{1.2}\frac{x^3}{n+2}\frac{d^2v}{dx^2} - \&c.\right);$$

or if $n = 1$,

$$\int dx\,(v) = xv - \frac{x^2}{1.2}\frac{dv}{dx} + \frac{x^3}{1.2.3}\frac{d^2v}{dx^2} - \&c.$$

which is the series of Bernoulli.

(11) In the theorem of Leibnitz let $v = \epsilon^{ax}$, then as $\frac{dv}{dx} = a\epsilon^{ax} = av$, we have $\frac{d'}{dx} = a$, and therefore

$$\left(\frac{d}{dx}\right)^n (\epsilon^{ax}u) = \left(a + \frac{d}{dx}\right)^n u \,.\, \epsilon^{ax};$$

$$\text{whence} \quad \left(a + \frac{d}{dx}\right)^n u = \epsilon^{-ax}\left(\frac{d}{dx}\right)^n \epsilon^{ax}u.$$

This result is of great use in the Integration of Linear Differential Equations.

(12) If we assume as before

$$\epsilon^{\frac{d}{dx}} = E;$$

we have $\qquad E^{nh}f(x) = f(x + nh).$

$$\text{Now } E^{nh} - 1 = \frac{E^{nh} - 1}{E^h - 1}(E^h - 1) = \frac{E^{nh} - 1}{E^h - 1}\left(\epsilon^{h\frac{d}{dx}} - 1\right);$$

or expanding the exponential

$$E^{nh}-1=\frac{E^{nh}-1}{E^{h}-1}\left\{h\frac{d}{dx}+\frac{h^2}{1.2}\left(\frac{d}{dx}\right)^2+\frac{h^3}{1.2.3}\left(\frac{d}{dx}\right)^3+\&c.\right\}$$

Apply these equivalent operations to $f(x)$, and indicate the successive differentials by accents affixed to the f; then

$$f(x+nh)-f(x)=\frac{E^{nh}-1}{E^{h}-1}\left\{hf'(x)+\frac{h^2}{1.2}f''(x)+\frac{h^3}{1.2.3}f'''(x)+\&c.\right\}$$

But $\dfrac{E^{nh}-1}{E^{h}-1}=E^{(n-1)h}+E^{(n-2)h}+E^{(n-3)h}+\&c.+1.$

Therefore, writing these in an inverse order and effecting the operations indicated, we find

$$\begin{aligned}f(x+nh)-f(x)&=h\left[f'(x)+f'(x+h)+\&c.+f'\{x+(n-1)h\}\right]\\&+\frac{h^2}{1.2}\left[f''(x)+f''(x+h)+\&c.+f''\{x+(n-1)\}h\right]\\&+\&c. \qquad\qquad +\&c.\end{aligned}$$

(13) Since we have

$$1+E^{h}+E^{2h}+\&c.+E^{(n-1)h}=\frac{E^{nh}-1}{E^{h}-1}=(E^{nh}-1)(\epsilon^{h\frac{d}{dx}}-1)^{-1},$$

we may expand the factor $(\epsilon^{h\frac{d}{dx}}-1)^{-1}$ by means of Bernoulli's Numbers; (See Chap. V. Sec. IV. Ex. 9) when it becomes

$$\frac{1}{h}\left(\frac{d}{dx}\right)^{-1}-\frac{1}{2}+\frac{B_1}{1.2}h\frac{d}{dx}-\frac{B_3}{1.2.3.4}h^3\left(\frac{d}{dx}\right)^3+\&c.$$

Applying these equivalent operations to $\dfrac{d}{dx}f(x)$ or $f'(x)$, multiplying by h and transposing, we have

$$\begin{aligned}(E^{nh}-1)f(x)&=h\left\{\tfrac{1}{2}+E^{h}+\&c.+E^{(n-1)h}+\tfrac{1}{2}E^{nh}\right\}f'(x)\\&-\frac{B_1}{1.2}h^2(E^{nh}-1)f''(x)+\frac{B_3}{1.2.3.4}h^4(E^{nh}-1)f^{iv}(x)-\&c.\end{aligned}$$

That is

$$\begin{aligned}f(x+nh)-f(x)=h\Big[\tfrac{1}{2}f'(x)+f'(x+h)+\&c.+\\ f\{x+(n-1)h\}+\tfrac{1}{2}f(x+nh)\Big]+\end{aligned}$$

$$- \frac{B_1}{1 \,.\, 2} h^2 \{f''(x+nh) - f''(x)\}$$
$$+ \frac{B_3}{1 \,.\, 2 \,.\, 3} h^4 \{f^{iv}(x+nh) - f^{iv}(x)\},$$
$$- \&c. \quad \&c.$$

The results in the two preceding examples are of great use in the approximate evaluation of definite integrals.

Poisson, *Mémoires de l' Institut*, 1823.

(14) Having given the transcendental equation

$$x = c\epsilon^{hx},$$

we can expand x in terms of c by means of the logarithmic method of solving equations: for the root of the preceding equation is the coefficient of $\frac{1}{x}$ in the expansion of $-\log\left(1 - \frac{c\epsilon^{hx}}{x}\right)$. This is easily found to be

$$c + \frac{2h}{1.2}c^2 + \frac{(3h)^2}{1 \,.\, 2 \,.\, 3}c^3 + \frac{(4h)^3}{1 \,.\, 2 \,.\, 3 \,.\, 4}c^4 + \&c.$$

Instead of x substitute $\frac{d}{dx}$; then $\epsilon^{h\frac{d}{dx}} = E^h$ and $c = \frac{d}{dx}E^{-h}$.

Hence we have

$$\frac{d}{dx} = \frac{d}{dx}E^{-h} + \frac{2h}{1 \,.\, 2}\left(\frac{d}{dx}\right)^2 E^{-2h} + \frac{(3h)^2}{1 \,.\, 2 \,.\, 3}\left(\frac{d}{dx}\right)^3 E^{-3h} + \&c.$$

Applying these equivalent operations to $\int dx f(x)$ we find

$$f(x) = f(x-h) + \frac{2h}{1 \,.\, 2}f'(x-2h) + \frac{(3h)^2}{1 \,.\, 2 \,.\, 3}f''(x-3h) + \&c.$$

This very remarkable theorem is given by Mr Murphy in the *Philosophical Transactions*.

(15) In a similar manner we may prove the more general theorem,

$$f(x) = f(x-nh) + nhf'\{x-(n+1)h\}$$
$$+ n(n+2)\frac{h^2}{1.2}f''\{x-(n+2)h\} + n(n+3)^2\frac{h^3}{1.2.3}f'''\{x-(n+3)h\} + \&c.$$

(16) We know by the Calculus of angular functions that

$$\frac{\pi}{4}\theta = \sin\theta - \frac{1}{3^2}\sin 3\theta + \frac{1}{5^2}\sin 5\theta - \&c.$$

Putting for the sines their exponential values and replacing $(-)^{\frac{1}{2}}\theta$ by $\left(h\frac{d}{dx}\right)$ we have

$$\frac{\pi}{2}h\frac{d}{dx} = \epsilon^{h\frac{d}{dx}} - \epsilon^{-h\frac{d}{dx}} - \frac{1}{3^2}\left(\epsilon^{3h\frac{d}{dx}} - \epsilon^{-3h\frac{d}{dx}}\right) + \&c.$$

Applying these equivalent operations to $\phi(x)$, we find

$$\frac{\pi}{2}h\frac{d}{dx}\phi(x) = \phi(x+h) - \phi(x-h)$$

$$-\frac{1}{3^2}\{\phi(x+3h) - \phi(x-3h)\} + \&c.$$

Français, *Annales des Mathématiques*, Vol. III. p. 252.

(17) . In the same manner from the equation

$$\tfrac{1}{2} = \cos\theta - \cos 2\theta + \cos 3\theta - \&c.,$$

we obtain the theorem,

$$\begin{aligned}\phi(x) = {} & \phi(x+h) - \phi(x+2h) + \phi(x+3h) - \&c.\\ & + \phi(x-h) - \phi(x-2h) + \phi(x-3h) - \&c.\end{aligned}$$

(18) Likewise by means of the equation

$$\frac{\theta}{2} = \sin\theta - \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta - \&c.$$

we find that

$$\begin{aligned}h\frac{d}{dx}\phi(x) = {} & \phi(x+h) - \frac{1}{2}\phi(x+2h) + \frac{1}{3}\phi(x+3h) - \&c.\\ & - \left\{\phi(x-h) - \frac{1}{2}\phi(x-2h) + \frac{1}{3}\phi(x-3h) - \&c.\right\}\end{aligned}$$

INTEGRAL CALCULUS.

CHAPTER I.

INTEGRATION OF FUNCTIONS OF ONE VARIABLE.

The fundamental formulæ to which all integrals are reduced are the following.

(*a*) $\int dx\, x^n = \dfrac{x^{n+1}}{n+1}$,

except when $n = -1$, in which case

(*b*) $\int \dfrac{dx}{x} = \log x$,

(*c*) $\int \dfrac{dx}{a^2+x^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a}$, (*d*) $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right)$,

(*e*) $\int \dfrac{dx}{(a^2-x^2)^{\frac{1}{2}}} = \sin^{-1}\dfrac{x}{a}$, and $\int \dfrac{-dx}{(a^2-x^2)^{\frac{1}{2}}} = \cos^{-1}\dfrac{x}{a}$,

(*f*) $\int \dfrac{dx}{(x^2 \pm a^2)^{\frac{1}{2}}} = \log\left\{\dfrac{(x^2 \pm a^2)^{\frac{1}{2}} + x}{a}\right\}$,

(*g*) $\int \dfrac{dx}{x(x^2-a^2)^{\frac{1}{2}}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a}$,

(*h*) $\int \dfrac{dx}{x(a^2 \pm x^2)^{\frac{1}{2}}} = \dfrac{1}{a}\log\left\{\dfrac{x}{(a^2 \pm x^2)^{\frac{1}{2}} + a}\right\}$,

(*i*) $\int dx\, a^x = \dfrac{a^x}{\log a}$, or $\int dx\, \epsilon^{ax} = \dfrac{\epsilon^{ax}}{a}$,

(*k*) $\int dx \sin mx = -\dfrac{1}{m}\cos mx$, and $\int dx \cos mx = \dfrac{1}{m}\sin mx$,

(*l*) $\int dx\, (\sec x)^2 = \tan x$.

By simple algebraic transformations we may frequently put an integral into a shape in which one or other of the preceding formulæ is at once applicable.

(1) $$\int \frac{dx\, x^{n-1}}{a+bx^n} = \frac{1}{nb}\int \frac{dx\, nbx^{n-1}}{a+bx^n} = \frac{1}{nb}\log(a+bx^n).$$

(2) $$\int \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = \int \frac{dx}{\{a^2-(a-x)^2\}^{\frac{1}{2}}} = -\int \frac{d(a-x)}{\{a^2-(a-x)^2\}^{\frac{1}{2}}}$$

$$= \cos^{-1}\frac{a-x}{a} = \operatorname{vers}^{-1}\frac{x}{a}.$$

(3) $$\int \frac{dx}{(2ax+x^2)^{\frac{1}{2}}} = \log\left\{\frac{(x^2+2ax)^{\frac{1}{2}}+x+a}{a}\right\}.$$

(4) $$\int \frac{dx\, x}{(a^4-x^4)^{\frac{1}{2}}} = \frac{1}{2}\int \frac{d(x^2)}{\{a^4-(x^2)^2\}^{\frac{1}{2}}} = \frac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$$

(5) $$\int \frac{dx\, x}{a^4+x^4} = \frac{1}{2}\int \frac{d(x^2)}{a^4+(x^2)^2} = \frac{1}{2a^2}\tan^{-1}\left(\frac{x^2}{a^2}\right).$$

(6) $$\int \frac{dx\, x}{\{(x^2-a^2)(b^2-x^2)\}^{\frac{1}{2}}} = \int \frac{d(x^2-a^2)^{\frac{1}{2}}}{\{b^2-a^2-(x^2-a^2)\}^{\frac{1}{2}}}$$

$$= \sin^{-1}\left(\frac{x^2-a^2}{b^2-a^2}\right)^{\frac{1}{2}},$$

$$\int \frac{dx}{a+bx+cx^2} = \frac{1}{c}\int \frac{dx}{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}},$$

which is integrated by (*c*) or by (*d*) according as $4ac-b^2>0$, or <0. Hence we have

(7) $$\int \frac{dx}{1+x+x^2} = \frac{2}{3^{\frac{1}{2}}}\tan^{-1}\left(\frac{2x+1}{3^{\frac{1}{2}}}\right).$$

(8) $$\int \frac{dx}{1+x-x^2} = \frac{1}{5^{\frac{1}{2}}}\log\left(\frac{2x-1+5^{\frac{1}{2}}}{2x-1-5^{\frac{1}{2}}}\right).$$

(9) $$\int \frac{dx}{1-2x+2x^2} = \tan^{-1}(2x-1).$$

(10) $$\int \frac{dx}{1+3x+2x^2} = \log\left\{\frac{2x+1}{2(x+1)}\right\}.$$

The integral $\int \frac{dx}{(a+bx \pm cx^2)^{\frac{1}{2}}}$ is reduced to

$$\frac{1}{c^{\frac{1}{2}}} \int \frac{dx}{\left\{\left(x+\frac{b}{2c}\right)^2 + \frac{4ac-b^2}{4c^2}\right\}^{\frac{1}{2}}}, \text{ or to } \frac{1}{c^{\frac{1}{2}}} \int \frac{dx}{\left\{\frac{4ac+b^2}{4c^2} - \left(x-\frac{b}{2c}\right)^2\right\}^{\frac{1}{2}}},$$

according as the upper or lower sign of c is taken; and these are of the forms (f) or (e) respectively. Hence

$$(11) \quad \int \frac{dx}{(1+x+x^2)^{\frac{1}{2}}} = \log\left\{2x+1+2(1+x+x^2)^{\frac{1}{2}}\right\}.$$

$$(12) \quad \int \frac{dx}{(1+2x-x^2)^{\frac{1}{2}}} = \sin^{-1}\left(\frac{x-1}{2^{\frac{1}{2}}}\right).$$

$$(13) \quad \int \frac{dx}{(x^2-x-1)^{\frac{1}{2}}} = \log\left\{2x-1+2(x^2-x-1)^{\frac{1}{2}}\right\}.$$

$$(14) \quad \int \frac{dx}{(1-x-x^2)^{\frac{1}{2}}} = \sin^{-1}\frac{2x+1}{5^{\frac{1}{2}}}.$$

The integral $\int \frac{dx\,(ax+b)}{x^2+px+q}$ may be split into

$$\left(b-\frac{ap}{2}\right)\int \frac{dx}{x^2+px+q} + \frac{a}{2}\int \frac{(2x+p)\,dx}{x^2+px+q},$$

the first of which is integrable by (c) and the second by (b). Hence

$$(15) \quad \int \frac{x\,dx}{a^2+2bx+x^2} = \log(a^2+2bx+x^2)^{\frac{1}{2}} - \frac{b}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{x+b}{(a^2-b^2)^{\frac{1}{2}}}\right\}.$$

$$(16) \quad \int \frac{(2x-1)\,dx}{x^2+2x+3} = \log(x^2+2x+3) - \frac{3}{2^{\frac{1}{2}}}\tan^{-1}\frac{x+1}{2^{\frac{1}{2}}}.$$

$$(17) \quad \int \frac{(1-x\cos\theta)\,dx}{1-2x\cos\theta+x^2} = \sin\theta\tan^{-1}\left(\frac{x-\cos\theta}{\sin\theta}\right) - \cos\theta\log(1-2x\cos\theta+x^2)^{\frac{1}{2}}.$$

In this example the numerator may be readily split by observing that $1 = \cos^2\theta + \sin^2\theta$.

$$(18)\quad \int \frac{(1-\frac{1}{2}x)\,dx}{1-x+x^2} = \frac{1}{2.3^{\frac{1}{2}}}\tan^{-1}\left(\frac{2x-1}{3^{\frac{1}{2}}}\right) - \log(1-x+x^2)^{\frac{1}{4}}.$$

By multiplying the numerator and denominator of a fraction by the same quantity it may frequently be split into integrable parts or reduced to an integrable shape.

$$(19)\quad \int dx\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \int dx\frac{1+x}{(1-x^2)^{\frac{1}{2}}} = \sin^{-1}x - (1-x^2)^{\frac{1}{2}}.$$

$$(20)\quad \int dx\frac{(x^2-a^2)^{\frac{1}{2}}}{x} = (x^2-a^2)^{\frac{1}{2}} - a\sec^{-1}\frac{x}{a}.$$

$$(21)\quad \int dx\,\frac{(x^2+a^2)^{\frac{1}{2}}}{x} = (x^2+a^2)^{\frac{1}{2}} + a\log\left\{\frac{x}{(x^2+a^2)^{\frac{1}{2}}+a}\right\}.$$

$$(22)\quad \int\frac{dx\,(x+a)^{\frac{1}{2}}}{x\,(x-a)^{\frac{1}{2}}} = \sec^{-1}\frac{x}{a} + \log\left\{\frac{x+(x^2-a^2)^{\frac{1}{2}}}{a}\right\}.$$

$$(23)\quad \int\frac{dx}{(x+a)^{\frac{1}{2}}+(x+b)^{\frac{1}{2}}} = \frac{2}{3(a-b)}\left\{(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right\}.$$

$$(24)\quad \int\frac{dx}{(1-x^2)^{\frac{3}{2}}} = \int\frac{x^{-3}\,dx}{(x^{-2}-1)^{\frac{3}{2}}} = \frac{x}{(1-x^2)^{\frac{1}{2}}}.$$

$$(25)\quad \int\frac{dx}{(a+bx^2)^{\frac{3}{2}}} = \frac{x}{a\,(a+bx^2)^{\frac{1}{2}}}.$$

$$(26)\quad \int\frac{dx}{x^2\,(1-x^2)^{\frac{1}{2}}} = \int\frac{x^{-3}\,dx}{(x^{-2}-1)^{\frac{1}{2}}} = -\frac{(1-x^2)^{\frac{1}{2}}}{x}.$$

$$(27)\quad \int\frac{dx}{(a+bx^n)^{\frac{n+1}{n}}} = \frac{x}{a\,(a+bx^n)^{\frac{1}{n}}}.$$

$$(28)\quad \int dx\,(a^2-x^2)^{\frac{1}{2}} = a^2\int\frac{dx}{(a^2-x^2)^{\frac{1}{2}}} - \int\frac{dx\,x^2}{(a^2-x^2)^{\frac{1}{2}}}$$

$$= \tfrac{1}{2}a^2\sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\,(a^2-x^2)^{\frac{1}{2}}.$$

(29) $$\int \frac{dx}{x(a+bx+cx^2)^{\frac{1}{2}}} = \int \frac{dx\,x^{-2}}{(ax^{-2}+bx^{-1}+c)^{\frac{1}{2}}}$$

$$= -\int \frac{d(x^{-1})}{(ax^{-2}+bx^{-1}+c)^{\frac{1}{2}}},$$

which is of the same form as $\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}$. Therefore

(30) $$\int \frac{dx}{x(1+x+x^2)^{\frac{1}{2}}} = \log \left\{ \frac{x}{2+x+2(1+x+x^2)^{\frac{1}{2}}} \right\}.$$

(31) $$\int \frac{dx}{x(x^2+2x-1)^{\frac{1}{2}}} = \sin^{-1}\left(\frac{x-1}{2^{\frac{1}{2}}x}\right).$$

(32) $$\int \frac{dx}{(a+bx+cx^2)^{\frac{3}{2}}} = \frac{2(2cx+b)}{(4ac-b^2)(a+bx+cx^2)^{\frac{1}{2}}}.$$

(33) $$\int \frac{x\,dx}{(a+bx+cx^2)^{\frac{3}{2}}} = -\frac{2(2a+bx)}{(4ac-b^2)(a+bx+cx^2)^{\frac{1}{2}}}.$$

(34) The integral $\int dx \frac{\epsilon^x x}{(1+x)^2}$ can be split into

$$\int dx\, \epsilon^x \left\{ \frac{1}{1+x} - \frac{1}{(1+x)^2} \right\},$$

and as the second term within the brackets is the differential of the first, it is equivalent to $\int dx \frac{d}{dx} \epsilon^x . \frac{1}{1+x}$; and therefore

$$\int dx \frac{\epsilon^x x}{(1+x)^2} = \frac{\epsilon^x}{1+x}.$$

(35) In the same way we shall find

$$\int dx \frac{\epsilon^x (2-x^2)}{(1+x)^{\frac{1}{2}}(1-x)^{\frac{3}{2}}} = \epsilon^x \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}}.$$

(36) $$\int dx\, \epsilon^x \frac{x^2+1}{(x+1)^2} = \epsilon^x \left(\frac{x-1}{x+1}\right).$$

(37) $$\int \frac{dx}{1+\epsilon^x} = \int \frac{dx\,\epsilon^{-x}}{1+\epsilon^{-x}} = \log\left(\frac{\epsilon^x}{1+\epsilon^x}\right).$$

(38) $\int dx \tan x = \int dx \frac{\sin x}{\cos x} = -\log(\cos x) = \log(\sec x).$

(39) $\int dx \cot x = \int dx \frac{\cos x}{\sin x} = \log(\sin x).$

(40) Since $\frac{d \tan x}{dx} = (\sec x)^2,$

$$\int \frac{dx}{\sin x \cos x} = \int dx \frac{(\sec x)^2}{\tan x} = \log(\tan x).$$

(41) Hence also as $\sin x = 2 \sin \frac{1}{2} x \cos \frac{1}{2} x$, we have

$$\int \frac{dx}{\sin x} = \log\left(\tan \frac{x}{2}\right), \text{ and as}$$

$$\cos x = \sin(\tfrac{1}{2}\pi - x) = \sin(\tfrac{1}{2}\pi + x),$$

$$\int \frac{dx}{\cos x} = \log\left\{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right\}.$$

(42) As $1 + (\tan x)^2 = (\sec x)^2$,

$$\int dx (\tan x)^2 = \int dx \left\{(\sec x)^2 - 1\right\} = \tan x - x.$$

(43) As $(\sin x)^2 + (\cos x)^2 = 1$,

$$\int \frac{dx}{(\sin x)^2 (\cos x)^2} = \int dx \frac{(\sin x)^2 + (\cos x)^2}{(\sin x)^2 (\cos x)^2}$$

$$= \int dx \left\{\frac{1}{(\cos x)^2} + \frac{1}{(\sin x)^2}\right\} = \tan x - \cot x = -2 \cot 2x.$$

(44) $\int \frac{dx}{a(1 + \cos x)} = \frac{1}{a} \int \frac{d(\frac{1}{2}x)}{(\cos \frac{1}{2}x)^2} = \frac{1}{a} \tan \frac{1}{2}x.$

(45) $\int \frac{dx \sin x}{a + b \cos x} = -\frac{1}{b} \log(a + b \cos x).$

The integral $\int \frac{dx}{a + b \cos x}$ may be reduced to the form (*c*); for as

$$\cos x = (\cos \tfrac{1}{2}x)^2 - (\sin \tfrac{1}{2}x)^2, \text{ and } 1 = (\cos \tfrac{1}{2}x)^2 + (\sin \tfrac{1}{2}x)^2,$$

it is equivalent to

$$\int \frac{dx}{(a+b)(\cos\frac{1}{2}x)^2+(a-b)(\sin\frac{1}{2}x)^2} = \int dx \frac{(\sec\frac{1}{2}x)^2}{a+b+(a-b)(\tan\frac{1}{2}x)^2};$$

which if $\tan\frac{1}{2}x = z$ takes the form

$$2\int \frac{dz}{a+b+(a-b)\,z^2},$$

and may therefore be integrated by (c). The result is

$$\int \frac{dx}{a+b\cos x} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\frac{1}{2}x\right\}$$

$$= \frac{1}{(a^2-b^2)^{\frac{1}{2}}}\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right);$$

$$\text{or} \int \frac{dx}{a+b\cos x} = \frac{1}{(b^2-a^2)^{\frac{1}{2}}}\log\left\{\frac{(b+a)^{\frac{1}{2}}+\tan\frac{1}{2}(b-a)^{\frac{1}{2}}x}{(b+a)^{\frac{1}{2}}-\tan\frac{1}{2}(b-a)^{\frac{1}{2}}x}\right\},$$

according as $a >$ or $< b$.

(46) $$\int \frac{dx}{2+\cos x} = \frac{2}{3^{\frac{1}{2}}}\tan^{-1}\left(\frac{\tan\frac{1}{2}x}{3^{\frac{1}{2}}}\right) = \frac{1}{3^{\frac{1}{2}}}\cos^{-1}\left(\frac{1+2\cos x}{2+\cos x}\right).$$

(47) $$\int \frac{dx}{1+2\cos x} = \frac{1}{3^{\frac{1}{2}}}\log\left(\frac{3^{\frac{1}{2}}+\tan\frac{1}{2}x}{3^{\frac{1}{2}}-\tan\frac{1}{2}x}\right).$$

In the same way we find

$$\int \frac{dx}{a+b\sin x} = \frac{2}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\frac{a\tan\frac{1}{2}x+b}{(a^2-b^2)^{\frac{1}{2}}}\right\} \text{ when } a > b,$$

$$\text{and } = \frac{1}{(b^2-a^2)^{\frac{1}{2}}}\log\left\{\frac{a\tan\frac{1}{2}x+b-(b^2-a^2)^{\frac{1}{2}}}{a\tan\frac{1}{2}x+b+(b^2-a^2)^{\frac{1}{2}}}\right\} \text{ when } a < b.$$

(48) $$\int \frac{dx}{5+4\sin x} = \frac{2}{3}\tan^{-1}\left(\frac{5\tan\frac{1}{2}x+4}{3}\right).$$

(49) $$\int \frac{dx}{4+5\sin x} = \frac{1}{3}\log\left(\frac{2\tan\frac{1}{2}x+1}{2\tan\frac{1}{2}x+4}\right).$$

(50) $$\int \frac{dx}{a(\cos x)^2+b(\sin x)^2} = \int \frac{dx\,(\sec x)^2}{a+b(\tan x)^2}$$

$$= \frac{1}{(ab)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{b}{a}\right)^{\frac{1}{2}}\tan x\right\}.$$

$$(51)\quad \int \frac{dx}{1+(\cos x)^2} = \frac{1}{2^{\frac{1}{2}}}\tan^{-1}\left(\frac{\tan x}{2^{\frac{1}{2}}}\right).$$

$$(52)\quad \int \frac{dx \sin x (\cos x)^2}{1+a^2(\cos x)^2} = \frac{1}{a^2}\int \frac{dx \sin x\{1+a^2(\cos x)^2-1\}}{1+a^2(\cos x)^2}$$

$$= \frac{1}{a^2}\int dx \sin x - \frac{1}{a^2}\int \frac{dx \sin x}{1+a^2(\cos x)^2}$$

$$= -\frac{\cos x}{a^2} + \frac{1}{a^3}\tan^{-1}(a\cos x).$$

$$(53)\quad \int \frac{dx}{a+b\tan x} = \int \frac{dx \cos x}{a\cos x + b\sin x}.$$

Adding and subtracting $\frac{a}{a^2+b^2}$ this becomes

$$\frac{b}{a^2+b^2}\int \frac{dx\,(b\cos x - a\sin x)}{a\cos x + b\sin x} + \frac{a}{a^2+b^2}\int dx;$$

and therefore

$$\int \frac{dx}{a+b\tan x} = \frac{1}{a^2+b^2}\{ax + b\log(a\cos x + b\sin x)\}.$$

$$(54)\quad \int \frac{dx \tan x}{\{a+b(\tan x)^2\}^{\frac{1}{2}}} = \int \frac{dx \sin x}{\{a(\cos x)^2 + b(\sin x)^2\}^{\frac{1}{2}}}$$

$$= \int \frac{dx \sin x}{\{b-(b-a)(\cos x)^2\}^{\frac{1}{2}}} = \frac{1}{(b-a)^{\frac{1}{2}}}\cos^{-1}\left\{\left(\frac{b-a}{b}\right)^{\frac{1}{2}}\cos x\right\}.$$

By means of the formulæ for expressing the products of sines and cosines of angles, in terms of sums and differences of sines and cosines of angles, we easily find

$$(55)\quad \int dx \sin mx \cos nx = -\frac{1}{2}\left\{\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right\}.$$

$$(56)\quad \int dx \sin mx \sin nx = -\frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right\}.$$

$$(57)\quad \int dx \cos mx \cos nx = \frac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\}.$$

$$(58)\quad \int dx \sin x \sin 2x \sin 3x = -\frac{1}{8}\left\{\cos 2x + \frac{\cos 4x}{2} - \frac{\cos 6x}{3}\right\}.$$

(59) $\int dx \cos x \cos 2x \cos 3x = \frac{1}{4}\left\{\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + x\right\}.$

(60) $\int dx \cos x \sin 2x \sin 3x = \frac{1}{4}\left\{x + \frac{\sin 2x}{2} - \frac{\sin 4x}{4} - \frac{\sin 6x}{6}\right\}.$

Integration by Parts.

Integration by parts often decomposes a function into an integrated part and one easily integrated. The general formula is

$$\int dx\, u\frac{dv}{dx} = uv - \int dx\, v\frac{du}{dx}.$$

(1) $\int dx\, x\epsilon^{ax} = \epsilon^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

(2) $\int dx \log x = x(\log x - 1).$

(3) $\int dx\, x\log x = \frac{x^2}{2}(\log x - \frac{1}{2}).$

(4) $\int dx\, x^n \log x = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$

(5) $\int dx \sin^{-1} x = x\sin^{-1} x + (1 - x^2)^{\frac{1}{2}}.$

(6) $\int \frac{dx}{(1 - x^2)^{\frac{1}{2}}} \sin^{-1} x = \frac{x\sin^{-1} x}{(1 - x^2)^{\frac{1}{2}}} + \log(1 - x^2)^{\frac{1}{2}}.$

(7) $\int \frac{dx\, x\sin^{-1} x}{(1 - x^2)^{\frac{1}{2}}} = x - (1 - x^2)^{\frac{1}{2}} \sin^{-1} x.$

(8) $\int dx \sin^{-1}\left(\frac{x}{a + x}\right)^{\frac{1}{2}} = (x + a)\sin^{-1}\left(\frac{x}{a + x}\right)^{\frac{1}{2}} - (ax)^{\frac{1}{2}}.$

(9) $\int dx\, x\sin^{-1}\frac{1}{2}\left(\frac{2a - x}{a}\right)^{\frac{1}{2}} = \frac{x^2}{2}\sin^{-1}\frac{1}{2}\left(\frac{2a - x}{a}\right)^{\frac{1}{2}} + \frac{1}{4}\int \frac{x^2 dx}{(4a^2 - x^2)^{\frac{1}{2}}}$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{2a - x}{a}\right)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{2a} - \frac{x}{8}(4a^2 - x^2)^{\frac{1}{2}}.$$

(51) $$\int \frac{dx}{1+(\cos x)^2} = \frac{1}{2^{\frac{1}{2}}} \tan^{-1}\left(\frac{\tan x}{2^{\frac{1}{2}}}\right).$$

(52) $$\int \frac{dx \sin x (\cos x)^2}{1+a^2(\cos x)^2} = \frac{1}{a^2}\int \frac{dx \sin x \{1+a^2(\cos x)^2-1\}}{1+a^2(\cos x)^2}$$

$$= \frac{1}{a^2}\int dx \sin x - \frac{1}{a^2}\int \frac{dx \sin x}{1+a^2(\cos x)^2}$$

$$= -\frac{\cos x}{a^2} + \frac{1}{a^3}\tan^{-1}(a\cos x).$$

(53) $$\int \frac{dx}{a+b\tan x} = \int \frac{dx \cos x}{a\cos x + b\sin x}.$$

Adding and subtracting $\frac{a}{a^2+b^2}$ this becomes

$$\frac{b}{a^2+b^2}\int \frac{dx\,(b\cos x - a\sin x)}{a\cos x + b\sin x} + \frac{a}{a^2+b^2}\int dx;$$

and therefore

$$\int \frac{dx}{a+b\tan x} = \frac{1}{a^2+b^2}\{ax + b\log(a\cos x + b\sin x)\}.$$

(54) $$\int \frac{dx \tan x}{\{a+b(\tan x)^2\}^{\frac{1}{2}}} = \int \frac{dx \sin x}{\{a(\cos x)^2 + b(\sin x)^2\}^{\frac{1}{2}}}$$

$$= \int \frac{dx \sin x}{\{b-(b-a)(\cos x)^2\}^{\frac{1}{2}}} = \frac{1}{(b-a)^{\frac{1}{2}}}\cos^{-1}\left\{\left(\frac{b-a}{b}\right)^{\frac{1}{2}}\cos x\right\}.$$

By means of the formulæ for expressing the products of sines and cosines of angles, in terms of sums and differences of sines and cosines of angles, we easily find

(55) $$\int dx \sin mx \cos nx = -\tfrac{1}{2}\left\{\frac{\cos(m+n)x}{m+n} + \frac{\cos(m-n)x}{m-n}\right\}.$$

(56) $$\int dx \sin mx \sin nx = -\tfrac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right\}.$$

(57) $$\int dx \cos mx \cos nx = \tfrac{1}{2}\left\{\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right\}.$$

(58) $$\int dx \sin x \sin 2x \sin 3x = -\tfrac{1}{8}\left\{\cos 2x + \frac{\cos 4x}{2} - \frac{\cos 6x}{3}\right\}.$$

(59) $\int dx \cos x \cos 2x \cos 3x = \frac{1}{4}\left\{\frac{\sin 6x}{6} + \frac{\sin 4x}{4} + \frac{\sin 2x}{2} + x\right\}.$

(60) $\int dx \cos x \sin 2x \sin 3x = \frac{1}{4}\left\{x + \frac{\sin 2x}{2} - \frac{\sin 4x}{4} - \frac{\sin 6x}{6}\right\}.$

Integration by Parts.

Integration by parts often decomposes a function into an integrated part and one easily integrated. The general formula is

$$\int dx\, u\frac{dv}{dx} = uv - \int dx\, v\frac{du}{dx}.$$

(1) $\int dx\, x\epsilon^{ax} = \epsilon^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

(2) $\int dx \log x = x(\log x - 1).$

(3) $\int dx\, x\log x = \frac{x^2}{2}(\log x - \frac{1}{2}).$

(4) $\int dx\, x^n \log x = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right).$

(5) $\int dx \sin^{-1}x = x\sin^{-1}x + (1 - x^2)^{\frac{1}{2}}.$

(6) $\int \frac{dx}{(1 - x^2)^{\frac{3}{2}}} \sin^{-1}x = \frac{x\sin^{-1}x}{(1 - x^2)^{\frac{1}{2}}} + \log(1 - x^2)^{\frac{1}{2}}.$

(7) $\int \frac{dx\, x\sin^{-1}x}{(1 - x^2)^{\frac{1}{2}}} = x - (1 - x^2)^{\frac{1}{2}} \sin^{-1}x.$

(8) $\int dx \sin^{-1}\left(\frac{x}{a + x}\right)^{\frac{1}{2}} = (x + a)\sin^{-1}\left(\frac{x}{a + x}\right)^{\frac{1}{2}} - (ax)^{\frac{1}{2}}.$

(9) $\int dx\, x\sin^{-1}\frac{1}{2}\left(\frac{2a-x}{a}\right)^{\frac{1}{2}} = \frac{x^2}{2}\sin^{-1}\frac{1}{2}\left(\frac{2a-x}{a}\right)^{\frac{1}{2}} + \frac{1}{4}\int\frac{x^2dx}{(4a^2-x^2)^{\frac{1}{2}}}$

$$= \frac{x^2}{2}\sin^{-1}\left(\frac{2a - x}{a}\right)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{2a} - \frac{x}{8}(4a^2 - x^2)^{\frac{1}{2}}.$$

(10) $\int dx \tan^{-1}x = x \tan^{-1}x - \log(1+x^2)^{\frac{1}{2}}.$

(11) $\int \frac{dx\, x^2}{1+x^2} \tan^{-1}x = (x - \frac{1}{2}\tan^{-1}x)\tan^{-1}x - \log(1+x^2)^{\frac{1}{2}}.$

By two integrations by parts we find

(12) $\int \frac{dx\, \epsilon^{a\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}} = \frac{\epsilon^{a\tan^{-1}x}(a+x)}{(1+a^2)(1+x^2)^{\frac{1}{2}}}.$

(13) $\int \frac{dx\, x\, \epsilon^{a\tan^{-1}x}}{(1+x^2)^{\frac{3}{2}}} = \frac{\epsilon^{a\tan^{-1}x}(ax-1)}{(1+a^2)(1+x^2)^{\frac{1}{2}}}.$

Also by a double integration by parts we obtain

(14) $\int dx\, \epsilon^{ax} \cos nx = \epsilon^{ax} \frac{(a\cos nx + n\sin nx)}{a^2+n^2}.$

(15) $\int dx\, \epsilon^{ax} \sin nx = \epsilon^{ax} \frac{(a\sin nx - n\cos nx)}{a^2+n^2}.$

On comparing these expressions with the formulæ in Ex. (10) of Chap. II. Sec. 1, of the Diff. Cal. it will be seen that they may be deduced from the latter by making $r = -1$.

Rational Fractions.

If $\frac{U}{V}$ be a rational fraction, in which the numerator is of lower dimensions than the denominator, it may always be decomposed into a sum of simpler fractions differing according to the form of V.

V may consist of factors of the forms

I. $x - a$, II. $(x-a)^n$,

III. $x^2 + ax + b$, IV. $(x^2 + ax + b)^n$.

I. To every factor of the form $x - a$ corresponds a partial fraction of the form $\frac{M}{x-a}$, where

$$M = \frac{U}{\frac{dV}{dx}} \text{ when } x = a.$$

II. To every factor of the form $(x - a)^n$ corresponds a series of partial fractions of the form

$$\frac{M}{(x-a)^n} + \frac{M_1}{(x-a)^{n-1}} + \&c. + \frac{M_{n-1}}{x-a}.$$

Any one of the coefficients as M_p is given by the equation

$$M_p = \frac{1}{1 \,.\, 2 \ldots p} \left(\frac{d}{dx}\right)^p \left(\frac{U}{Q}\right) \text{ when } x = a,$$

where $Q = \dfrac{V}{(x-a)^n}$.

III. To every factor of the form $x^2 + ax + b$ corresponds a fraction $\dfrac{Mx + N}{x^2 + ax + b}$. To determine the constants M and N, the expression

$$(2x + a) - (Mx + N) \frac{dV}{dx} = 0$$

is reduced by successive substitutions of $-(ax + b)$ for x^2 to the form

$$Ax + B = 0,$$

and from the conditions $A = 0$, $B = 0$, M and N are found.

IV. To every factor of the form $(x^2 + ax + b)^n$ corresponds a series of fractions of the form

$$\frac{Mx + N}{(x^2 + ax + b)^n} + \frac{M_1 x + N_1}{(x^2 + ax + b)^{n-1}} + \&c. \frac{M_{n-1}x + N_{n-1}}{x^2 + ax + b}.$$

To determine M and N let $V = Q\,(x^2 + ax + b)^n$; then if by the successive substitutions of $-(ax + b)$ for x^2 the equation

$$U - (Mx + N)\,Q = 0$$

be reduced to the form

$$Ax + B = 0,$$

the equations $A = 0$, $B = 0$ are conditions for finding M and N. If now we put

$$\frac{U - (Mx + N)\,Q}{x^2 + ax + b} = U_1,$$

where U_1 is necessarily an integral function, we can, from the equation

$$U_1 - (M_1 x + N_1)\, Q = 0,$$

determine M_1 and N_1 as before, and so in succession for all the other partial fractions.

The fraction having been thus, by one or other of these methods, decomposed into a sum of simpler fractions, each of them may be integrated separately by known processes, and so the whole integral is found.

If the partial fraction be of the form $\dfrac{M}{x-a}$, we have

$$M\int\frac{dx}{x-a} = M\log(x-a) = \log(x-a)^M.$$

If the partial fraction be of the form $\dfrac{M}{(x-a)^r}$, we have

$$M\int\frac{dx}{(x-a)^r} = -\frac{M}{(r-1)}\,\frac{1}{(x-a)^{r-1}}.$$

If the partial fraction be of the form $\dfrac{Mx+N}{(x-a)^2+\beta^2}$, we have

$$\int\frac{dx\,(Mx+N)}{(x-a)^2+\beta^2} = M\log\left\{(x-a)^2+\beta^2\right\}^{\frac{1}{2}} + \frac{Ma+N}{\beta}\tan^{-1}\left(\frac{x-a}{\beta}\right).$$

If the partial fraction be of the form $\dfrac{Mx+N}{\left\{(x-a)^2+\beta^2\right\}^r}$,

$$\int\frac{dx\,(Mx+N)}{\left\{(x-a)^2+\beta^2\right\}^r} = \frac{M}{2(r-1)}\,\frac{1}{\left\{(x-a)^2+\beta^2\right\}^{r-1}} + (Ma+N)\int\frac{dx}{\left\{(x-a)^2+\beta^2\right\}^r}.$$

The expression for the last integral will be found in the following chapter on formulæ of reduction.

(1) Let $\dfrac{U}{V} = \dfrac{2x+3}{x^3+x^2-2x}$.

In this case the factors of V are x, $x-1$, $x+2$, and as

$$\frac{dV}{dx} = 3x^2+2x-2,$$

the coefficient corresponding to x is $-\dfrac{3}{2}$,

.... $x-1$ is $\dfrac{5}{3}$,

.... $x+2$ is $-\dfrac{1}{6}$.

Hence $$\frac{U}{V}=\frac{5}{3}\,\frac{1}{x-1}-\frac{1}{6}\,\frac{1}{x+2}-\frac{3}{2}\,\frac{1}{x},$$

and $$\int\frac{(2x+3)\,dx}{x^3+x^2-2x}=\log\frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}}\,(x+2)^{\frac{1}{6}}}.$$

(2) Let $\dfrac{U}{V}=\dfrac{x-1}{x^2+6x+8}$, then

$$\int\frac{(x-1)\,dx}{x^2+6x+8}=\log\frac{(x+4)^{\frac{5}{2}}}{(x+2)^{\frac{3}{2}}}.$$

(3) Let $\dfrac{U}{V}=\dfrac{x^2-x+2}{x^4-5x^2+4}$, then

$$\int\frac{(x^2-x+2)\,dx}{x^4-5x^2+4}=\log\left\{\frac{(x+1)^2(x-2)}{(x-1)\,(x+2)^2}\right\}^{\frac{1}{3}}.$$

(4) Let the fraction be of the form

$$\frac{x^r}{(x-a_1)\,(x-a_2)\ldots(x-a_n)},$$

where $r<n$; then

$$\int\frac{x^r\,dx}{(x-a_1)\,(x-a_2)\ldots(x-a_n)}=\frac{a_1{}^r\log(x-a_1)}{(a_1-a_2)\,(a_1-a_3)\ldots(a_1-a_n)}$$
$$+\frac{a_2{}^r\log(x-a_2)}{(a_2-a_1)\,(a_2-a_3)\ldots(a_2-a_n)}+\ldots+\frac{a_n{}^r\log(x-a_n)}{(a_n-a_1)\,(a_n-a_2)\ldots(a_n-a_{n-1})}.$$

(5) Let $\dfrac{U}{V}=\dfrac{1}{x^3-x^2-x+1}$.

Here the denominator contains two equal factors $(x-1)^2$, and the partial fractions arising from these equal roots are

$$\frac{1}{2}\,\frac{1}{(x-1)^2}-\frac{1}{4}\,\frac{1}{x-1},$$

and the fraction corresponding to the other factor $(x + 1)$ is

$$\tfrac{1}{4}\,\frac{1}{x+1}.$$

Hence $$\int\frac{dx}{x^3 - x^2 - x + 1} = \log\left(\frac{x+1}{x-1}\right)^{\frac{1}{4}} - \tfrac{1}{2}\,\frac{1}{x-1}.$$

(6) If $$\frac{U}{V} = \frac{2x^3 + 7x^2 + 6x + 2}{x^4 + 3x^3 + 2x^2}.$$

The roots of the denominator are -2, -1 and two equal to 0.

Therefore $$\int dx\,\frac{2x^3+7x^2+6x+2}{x^4 + 3x^3 + 2x^2} = \log\left\{x\,(x+1)\left(\frac{x}{x+2}\right)^{\frac{1}{2}}\right\} - \frac{1}{x}.$$

(7) If $$\frac{U}{V} = \frac{x^2}{x^3 + 5x^2 + 8x + 4},$$

$$\int\frac{x^2\,dx}{x^3 + 5x^2 + 8x + 4} = \frac{4}{x+2} + \log(x+1).$$

(8) Let $$\frac{U}{V} = \frac{1}{x^n\,(x-1)^n},$$ n being even;

$$\begin{aligned}\frac{U}{V} = \frac{1}{x^n} + \frac{1}{(1-x)^n} &+ n\left\{\frac{1}{x^{n-1}} + \frac{1}{(1-x)^{n-1}}\right\}\\ &+ \frac{n\,(n+1)}{1\,.\,2}\left\{\frac{1}{x^{n-2}} + \frac{1}{(1-x)^{n-2}}\right\} + \&c.\\ &+ \frac{n\,(n+1)\ldots 2\,(n-1)}{1\,.\,2\ldots(n-1)}\left\{\frac{1}{x} + \frac{1}{1-x}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int\frac{dx}{x^n\,(x-1)^n} = \frac{1}{n-1}\left\{\frac{1}{(1-x)^{n-1}} - \frac{1}{x^{n-1}}\right\} &+ \frac{n}{n-2}\left\{\frac{1}{(1-x)^{n-2}} - \frac{1}{x^{n-2}}\right\} + \&c.\\ &+ \frac{n\,(n+1)\ldots 2\,(n-1)}{1\,.\,2\ldots(n-1)}\log\left(\frac{x}{1-x}\right).\end{aligned}$$

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(9) Let $\dfrac{U}{V} = \dfrac{x^2}{x^4 + x^2 - 2}$.

The factors of V are $x + 1$, $x - 1$, and $x^2 + 2$. Hence

$$\int \frac{x^2\,dx}{x^4 + x^2 - 2} = \frac{1}{6}\log\left(\frac{x-1}{x+1}\right) + \frac{2^{\frac{1}{2}}}{3}\tan^{-1}\frac{x}{2^{\frac{1}{2}}}.$$

(10) Let $\dfrac{U}{V} = \dfrac{x^3}{x^4 + 3x^2 + 2}$;

$$\int \frac{x^3\,dx}{x^4 + 3x^2 + 2} = \log\left\{\frac{x^2 + 2}{(x^2 + 1)^{\frac{1}{2}}}\right\}.$$

(11) Let $\dfrac{U}{V} = \dfrac{x^2}{x^3 + x^2 + x + 1}$;

$$\int \frac{x^2\,dx}{x^3 + x^2 + x + 1} = \tfrac{1}{2}\log\left\{(x + 1)(x^2 + 1)^{\frac{1}{2}}\right\} - \tfrac{1}{2}\tan^{-1}x.$$

(12) Let $\dfrac{U}{V} = \dfrac{3x^2 + x - 2}{(x - 1)^3(x^2 + 1)}$;

$$\int \frac{(3x^2 + x - 2)\,dx}{(x - 1)^3(x^2 + 1)} = \frac{3}{4}\log\frac{x^2 + 1}{(x - 1)^2} - \tan^{-1}x$$

$$- \frac{1}{2}\frac{1}{(x - 1)^2} - \frac{5}{2}\frac{1}{x - 1}.$$

(13) Let $\dfrac{U}{V} = \dfrac{1}{x^5 - 3x^3 - 4x^2}$; then

$$\int \frac{dx}{x^5 + 2x^3 + 3x^2} = \frac{1}{90}\log\frac{(x^2 - x + 3)(x + 1)^{18}}{x^{20}} - \frac{1}{3x}$$

$$- \frac{13}{45\,11^{\frac{1}{2}}}\tan^{-1}\frac{2x - 1}{11^{\frac{1}{2}}}.$$

(14) Let $\dfrac{U}{V} = \dfrac{1}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1}$.

Here there are in the denominator two equal quadratic factors $(x^2 + 1)$; the fractions arising from them are

$$-\frac{1}{2}\frac{x - 1}{(x^2 + 1)^2} - \frac{1}{4}\frac{x - 1}{x^2 + 1}.$$

Hence

$$\int \frac{dx}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} = \frac{1}{4}\frac{x+1}{x^2+1} + \frac{1}{2}\tan^{-1} x$$

$$+ \frac{1}{4}\log\frac{x+1}{(x^2+1)^{\frac{1}{2}}}.$$

(15) Let $\dfrac{U}{V} = \dfrac{x^2 + 3x - 2}{(x^2 - x + 1)^2 (x-1)^2}$; then

$$\int \frac{U}{V} dx = \frac{-(5x-7)}{3(x^2 - x + 1)} - \frac{2}{x-1} - \frac{25}{3^{\frac{3}{2}}}\tan^{-1}\frac{2x-1}{3^{\frac{1}{2}}}$$

$$- \log\frac{(x^2 - x + 1)^{\frac{1}{2}}}{x-1}.$$

(16) $\displaystyle\int \frac{dx}{1+x^3} = \frac{1}{3}\log\frac{(x+1)}{(x^2 - x + 1)^{\frac{1}{2}}} + \frac{1}{3^{\frac{1}{2}}}\tan^{-1}\left(\frac{2x-1}{3^{\frac{1}{2}}}\right).$

(17) $\displaystyle\int \frac{dx}{x(a + bx^3)} = \frac{1}{3a}\log\left(\frac{x^3}{a + bx^3}\right).$

(18) $\displaystyle\int \frac{dx}{x^4 (a + bx^3)} = -\frac{1}{3ax^3} + \frac{b}{3a^2}\log\left(\frac{a + bx^3}{x^3}\right).$

(19) $\displaystyle\int \frac{dx}{x(a + bx^3)^2} = \frac{1}{3a(a + bx^3)} - \frac{1}{3a^2}\log\left(\frac{a + bx^3}{x^3}\right).$

(20) $\displaystyle\int \frac{dx}{1+x^4} = \frac{1}{2^{\frac{5}{2}}}\log\left(\frac{1 + 2^{\frac{1}{2}}x + x^2}{1 - 2^{\frac{1}{2}}x + x^2}\right) + \frac{1}{2^{\frac{3}{2}}}\tan^{-1}\left(\frac{2^{\frac{1}{2}}x}{1 - x^2}\right).$

(21) $\displaystyle\int \frac{dx}{1-x^4} = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} + \tfrac{1}{2}\tan^{-1} x.$

(22) $\displaystyle\int \frac{x^2\,dx}{1-x^4} = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} - \tfrac{1}{2}\tan^{-1} x.$

(23) $\displaystyle\int \frac{dx}{x(a + bx^4)} = -\frac{1}{4a}\log\left(\frac{a + bx^4}{x^4}\right).$

(24) $\displaystyle\int \frac{dx}{1-x^6} = \frac{1}{6}\log\frac{1+x}{1-x}\left(\frac{1 + x + x^2}{1 - x + x^2}\right)^{\frac{1}{2}}$

$$+ \frac{1}{2\,.\,3^{\frac{1}{2}}}\tan^{-1}\left(\frac{3^{\frac{1}{2}}x}{1 - x^2}\right).$$

(25) $$\int\frac{x^2\,dx}{a^6+x^6}=\frac{1}{3a^3}\tan^{-1}\left(\frac{x}{a}\right)^3.$$

(26) $$\int\frac{x^2\,dx}{1-x^6}=\frac{1}{6}\log\left(\frac{1+x^3}{1-x^3}\right).$$

Rationalization.

Integrals of the form $\int dx\,x^m(a+bx^n)^{\frac{p}{q}}$ can be rationalized, when $\dfrac{m+1}{n}$ is an integer, by assuming $a+bx^n=z^q$, and, when $\dfrac{m+1}{n}+\dfrac{p}{q}$ is an integer, by assuming $a+bx^n=x^n z^q$.

(1) $$\int\frac{x\,dx}{(a+bx)^{\frac{1}{2}}}=\frac{2}{3b^2}(a+bx)^{\frac{1}{2}}(bx-2a).$$

(2) $$\int\frac{x^3dx}{(x-1)^{\frac{1}{2}}}=2(x-1)^{\frac{1}{2}}\left\{\frac{(x-1)^3}{7}+\frac{3}{5}(x-1)^2+x\right\}.$$

(3) $$\int\frac{dx}{x\,(a+bx)^{\frac{1}{2}}}=\frac{1}{a^{\frac{1}{2}}}\log\left\{\frac{(a+bx)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(a+bx)^{\frac{1}{2}}+a^{\frac{1}{2}}}\right\}.$$

(4) $$\int\frac{dx}{x(bx-a)^{\frac{1}{2}}}=\frac{2}{a^{\frac{1}{2}}}\tan^{-1}\left(\frac{bx-a}{a}\right)^{\frac{1}{2}}=\frac{2}{a^{\frac{1}{2}}}\cos^{-1}\left(\frac{a}{bx}\right)^{\frac{1}{2}}.$$

(5) $$\int\frac{dx}{x^3(x-1)^{\frac{1}{2}}}=(x-1)^{\frac{1}{2}}\left(\frac{3x+2}{4x^2}\right)+\frac{3}{4}\cos^{-1}\left(\frac{1}{x}\right)^{\frac{1}{2}}.$$

(6) $$\int dx\,x^2(a^2+x^2)^{\frac{1}{2}}=2(a^2+x^2)^{\frac{3}{2}}\left\{\frac{(a^2+x^2)^2}{7}-\frac{2a^2(a^2+x^2)}{5}+\frac{a^4}{3}\right\}.$$

(7) $$\int\frac{dx\,x}{(a+bx)^{\frac{3}{2}}}=\frac{2}{b^2}\left\{\frac{2a+bx}{(a+bx)^{\frac{1}{2}}}\right\}.$$

(8) $$\int dx\,x(a+bx)^{\frac{3}{2}}=\frac{2(a+bx)^{\frac{5}{2}}}{b^2}\left(\frac{a+bx}{7}-\frac{a}{5}\right).$$

(9) $$\int\frac{dx\,x^2}{(a+bx)^{\frac{5}{2}}}=\frac{2}{3b^3}\,\frac{3(a+bx)^2+6a(a+bx)-a^2}{(a+bx)^{\frac{3}{2}}}.$$

Hence

$$\int \frac{dx}{x^5 + x^4 + 2x^3 + 2x^2 + x + 1} = \frac{1}{4}\frac{x+1}{x^2+1} + \frac{1}{2}\tan^{-1} x$$

$$+ \frac{1}{4}\log\frac{x+1}{(x^2+1)^{\frac{1}{2}}}.$$

(15) Let $\dfrac{U}{V} = \dfrac{x^2 + 3x - 2}{(x^2 - x + 1)^2 (x-1)^2}$; then

$$\int \frac{U}{V} dx = \frac{-(5x-7)}{3(x^2 - x + 1)} - \frac{2}{x-1} - \frac{25}{3^{\frac{3}{2}}}\tan^{-1}\frac{2x-1}{3^{\frac{1}{2}}}$$

$$- \log\frac{(x^2 - x + 1)^{\frac{1}{2}}}{x-1}.$$

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(21) $$\int \frac{dx}{1 - x^4} = \log\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}} + \frac{1}{2}\tan^{-1} x.$$

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(24) $$\int \frac{dx}{1 - x^6} = \frac{1}{6}\log\frac{1+x}{1-x}\left(\frac{1 + x + x^2}{1 - x + x^2}\right)^{\frac{1}{2}}$$

$$+ \frac{1}{2\,.\,3^{\frac{1}{2}}}\tan^{-1}\left(\frac{3^{\frac{1}{2}}x}{1 - x^2}\right).$$

(25) $\displaystyle\int\frac{x^2\,dx}{a^6+x^6}=\frac{1}{3a^3}\tan^{-1}\left(\frac{x}{a}\right)^3.$

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(1) $\displaystyle\int\frac{x\,dx}{(a+bx)^{\frac{1}{2}}}=\frac{2}{3b^2}(a+bx)^{\frac{1}{2}}(bx-2a).$

(2) $\displaystyle\int\frac{x^3\,dx}{(x-1)^{\frac{1}{2}}}=2(x-1)^{\frac{1}{2}}\left\{\frac{(x-1)^3}{7}+\frac{3}{5}(x-1)^2+x\right\}.$

(3) $\displaystyle\int\frac{dx}{x(a+bx)^{\frac{1}{2}}}=\frac{1}{a^{\frac{1}{2}}}\log\left\{\frac{(a+bx)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(a+bx)^{\frac{1}{2}}+a^{\frac{1}{2}}}\right\}.$

(4) $\displaystyle\int\frac{dx}{x(bx-a)^{\frac{1}{2}}}=\frac{2}{a^{\frac{1}{2}}}\tan^{-1}\left(\frac{bx-a}{a}\right)^{\frac{1}{2}}=\frac{2}{a^{\frac{1}{2}}}\cos^{-1}\left(\frac{a}{bx}\right)^{\frac{1}{2}}.$

(5) $\displaystyle\int\frac{dx}{x^3(x-1)^{\frac{1}{2}}}=(x-1)^{\frac{1}{2}}\left(\frac{3x+2}{4x^2}\right)+\frac{3}{4}\cos^{-1}\left(\frac{1}{x}\right)^{\frac{1}{2}}.$

(6) $\displaystyle\int dx\,x^2(a^2+x^2)^{\frac{1}{2}}=2(a^2+x^2)^{\frac{3}{2}}\left\{\frac{(a^2+x^2)^2}{7}-\frac{2a^2(a^2+x^2)}{5}+\frac{a^4}{3}\right\}.$

(7) $\displaystyle\int\frac{dx\,x}{(a+bx)^{\frac{3}{2}}}=\frac{2}{b^2}\left\{\frac{2a+bx}{(a+bx)^{\frac{1}{2}}}\right\}.$

(8) $\displaystyle\int dx\,x(a+bx)^{\frac{3}{2}}=\frac{2(a+bx)^{\frac{5}{2}}}{b^2}\left(\frac{a+bx}{7}-\frac{a}{5}\right).$

(9) $\displaystyle\int\frac{dx\,x^2}{(a+bx)^{\frac{5}{2}}}=\frac{2}{3b^3}\,\frac{3(a+bx)^2+6a(a+bx)-a^2}{(a+bx)^{\frac{3}{2}}}.$

(10) $$\int \frac{x\,dx}{(a+bx)^{\frac{1}{3}}} = \frac{3}{b^2}(a+bx)^{\frac{2}{3}}\left(\frac{a+bx}{5} - \frac{a}{2}\right).$$

(11) $$\int \frac{x^2\,dx}{(1+x)^{\frac{2}{3}}} = 3\,(1+x)^{\frac{1}{3}}\left\{\frac{(1+x)^2}{7} + \frac{1-x}{2}\right\}.$$

(12) $$\int dx\,x\,(a+x)^{\frac{1}{3}} = \frac{3\,(a+x)^{\frac{4}{3}}}{4}\left(\frac{4x-3}{7}\right).$$

(13) $$\int dx\,x^3\,(a+x)^{\frac{2}{3}} = 3\,(a+x)^{\frac{5}{3}}\left\{\frac{(a+x)^3}{14} - \frac{3a(a+x)^2}{11} + \frac{3a^2\,(a+x)}{8} - \frac{a^3}{5}\right\}.$$

(14) $$\int \frac{dx\,x^3}{(a+bx^2)^{\frac{1}{2}}} = \frac{bx^2-2a}{3b^2}(a+bx^2)^{\frac{1}{2}}.$$

(15) $$\int \frac{dx}{x^4\,(1+x^2)^{\frac{1}{2}}} = \frac{2x^2-1}{3x^3}(1+x^2)^{\frac{1}{2}}.$$

(16) $$\int \frac{dx\,x^6}{(1+x^2)^{\frac{1}{2}}} = (1+x^2)^{\frac{1}{2}}\left\{\frac{x^5}{6} - \frac{5x^3}{6.4} + \frac{5.3.x}{6.4.2}\right\}.$$

(17) $$\int \frac{dx\,x^3}{(a+bx^2)^{\frac{3}{2}}} = \frac{2a+bx^2}{b^2\,(a+bx^2)^{\frac{1}{2}}}.$$

(18) $$\int \frac{dx\,x^4}{(1-x^2)^{\frac{3}{2}}} = -\frac{x\,(x^2-3)}{2\,(1-x^2)^{\frac{1}{2}}} - \frac{3}{2}\sin^{-1}x.$$

(19) $$\int \frac{dx}{x^2\,(a+bx^2)^{\frac{3}{2}}} = -\frac{(a+2bx^2)}{a^2x\,(a+bx^2)^{\frac{1}{2}}}.$$

(20) $$\int \frac{dx}{(1+x^2)^{\frac{5}{2}}} = \frac{x\,(2x^2+3)}{3\,(1+x^2)^{\frac{3}{2}}}.$$

(21) $$\int \frac{dx\,x^2}{(a+bx^2)^{\frac{5}{2}}} = \frac{x^3}{3a\,(a+bx^2)^{\frac{3}{2}}}.$$

If an integral be a function of several fractional powers of x, it may be rationalized by assuming $x = z^r$, r being equal to the product of the denominators of the indices.

(22) $\int dx\,\dfrac{1-x^{\frac{1}{2}}}{1-x^{\frac{1}{3}}} = 6\int dz\,z^5\,\dfrac{1-z^3}{1-z^2}$,

by assuming $x = z^6$. This is equivalent to

$$6\int dz\left(z^5+z^4-z^3+z^2-z+1-\frac{1}{1+z}\right)$$

$$=6\left\{\frac{x^{\frac{7}{6}}}{7}+\frac{x^{\frac{5}{6}}}{5}-\frac{x^{\frac{2}{3}}}{4}+\frac{x^{\frac{1}{2}}}{3}-\frac{x^{\frac{1}{3}}}{2}+x^{\frac{1}{6}}-\log\left(1+x^{\frac{1}{6}}\right)\right\}.$$

(23) $\int dx\,\dfrac{1+x^{\frac{1}{4}}}{1+x^{\frac{1}{3}}}=\dfrac{x^{\frac{11}{12}}}{11}+\dfrac{x^{\frac{2}{3}}}{8}-\dfrac{x^{\frac{7}{12}}}{7}-\dfrac{x^{\frac{1}{3}}}{4}+\dfrac{x^{\frac{1}{4}}}{3}$

$$+\frac{1}{2^{\frac{5}{2}}}\left\{\log\left(\frac{x^{\frac{1}{6}}-2^{\frac{1}{2}}x^{\frac{1}{12}}+1}{x^{\frac{1}{6}}+2^{\frac{1}{2}}x^{\frac{1}{12}}+1}\right)+2\tan^{-1}\frac{2^{\frac{1}{2}}x^{\frac{1}{12}}}{1-x^{\frac{1}{6}}}\right\}.$$

When the integral involves also fractional powers of binomials, such as $a+bx$, it can be rationalized by assuming $a+bx=z^r$, r being the product of the denominators of the indices. If the binomials be of the form $a+bx^n$, they may be reduced to the preceding form by assuming $x^n=y$.

(24) Let the integral be $\int\dfrac{dx}{(1-x)(1+x)^{\frac{1}{2}}}$.

Assume $(1+x)^{-\frac{1}{2}}=z$, then

$$\int\frac{dx}{(1-x)(1+x)^{\frac{1}{2}}}=-2\int\frac{dz}{2z^2-1}=\frac{1}{2^{\frac{1}{2}}}\log\left(\frac{2^{\frac{1}{2}}z+1}{2^{\frac{1}{2}}z-1}\right)$$

$$=\frac{1}{2^{\frac{1}{2}}}\log\left\{\frac{2^{\frac{1}{2}}+(1+x)^{\frac{1}{2}}}{2^{\frac{1}{2}}-(1+x)^{\frac{1}{2}}}\right\}.$$

(25) $\int\dfrac{dx}{(2+x)(1+x)^{\frac{1}{2}}}=2\tan^{-1}(1+x)^{\frac{1}{2}}$.

(26) $\int\dfrac{dx}{(1+x)(1+x^2)^{\frac{1}{2}}}=\dfrac{1}{2^{\frac{1}{2}}}\log\left\{\dfrac{1-x-2^{\frac{1}{2}}(1+x^2)^{\frac{1}{2}}}{1+x}\right\}$.

(27) $\int\dfrac{dx}{(1+x^2)(1-x^2)^{\frac{1}{2}}}=\dfrac{1}{2^{\frac{1}{2}}}\tan^{-1}\left\{\dfrac{2^{\frac{1}{2}}x}{(1-x^2)^{\frac{1}{2}}}\right\}=\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}$.

(28) $\int\dfrac{dx}{(1-x^2)(1+x^2)^{\frac{1}{2}}}=\dfrac{1}{2^{\frac{1}{2}}}\log\left\{\dfrac{(1+x^2)^{\frac{1}{2}}+2^{\frac{1}{2}}x}{(1-x^2)^{\frac{1}{2}}}\right\}$.

(29) $$\int \frac{dx}{(1+x^2)(x^2-1)^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}}\log\left\{\frac{(x^2-1)^{\frac{1}{2}}+2^{\frac{1}{2}}x}{(1+x^2)^{\frac{1}{2}}}\right\}.$$

(30) $$\int \frac{dx}{(c+ex^2)(a+bx^2)^{\frac{1}{2}}} = \sin^{-1} x\left(\frac{ae-bc}{ac+aex^2}\right)^{\frac{1}{2}}, ae>bc$$

$$= \frac{1}{(bc^2-ace)^{\frac{1}{2}}}\log\frac{c(a+bx^2)^{\frac{1}{2}}+x(bc^2-ace)^{\frac{1}{2}}}{(c+ex^2)^{\frac{1}{2}}}, \quad ae<bc$$

$$= \frac{x}{c(a+bx^2)^{\frac{1}{2}}} \text{ if } ae=bc.$$

(31) $$\int \frac{dx}{x(a+bx^n)^{\frac{1}{2}}} = \frac{2}{na^{\frac{1}{2}}}\log\frac{(a+bx^n)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(bx^n)^{\frac{1}{2}}}.$$

(32) $$\int \frac{dx\,x^{n-1}}{(a-bx^{2n})^{\frac{1}{2}}} = \frac{1}{nb^{\frac{1}{2}}}\sec^{-1}\left\{\left(\frac{a}{b}\right)^{\frac{1}{2}}\frac{1}{x^n}\right\}.$$

If the function to be integrated involve $(a+bx\pm cx^2)^{\frac{1}{2}}$ it may be reduced to the preceding forms, as

$$(a+bx+cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}\left\{\left(x+\frac{b}{2c}\right)^2+\frac{4ac-b^2}{4c^2}\right\}^{\frac{1}{2}},$$

$$(a+bx-cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}\left\{\frac{4ac+b^2}{4c^2}-\left(x-\frac{b}{2c}\right)^2\right\}^{\frac{1}{2}}.$$

(33) $$\int \frac{dx}{(1+x)(1+x+x^2)^{\frac{1}{2}}} = \log\left\{\frac{1-x-2(1+x+x^2)^{\frac{1}{2}}}{1+x}\right\}.$$

(34) $$\int \frac{dx}{(1+x)(1+x-x^2)^{\frac{1}{2}}} = \tan^{-1}\left\{\frac{1+3x}{2(1+x-x^2)^{\frac{1}{2}}}\right\}.$$

(35) $$\int \frac{dx}{(1+x)(1-x-x^2)^{\frac{1}{2}}} = \log\left\{\frac{3+x-2(1-x-x^2)^{\frac{1}{2}}}{1+x}\right\}.$$

Various functions can be rationalized by assumptions for which no general rule can be given: familiarity with the transformations to which different substitutions lead is the best way of acquiring a knowledge of the most convenient assumption in particular cases.

(36) Let the integral be $\int dx\,\frac{\{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{m}{n}}}{(1+x^2)^{\frac{1}{2}}}$. By assuming $x+(1+x^2)^{\frac{1}{2}}=z^n$, the transformed integral becomes

$$n\int dz\,z^{m-1} = \frac{n}{m}z^m = \frac{n}{m}\{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{m}{n}}.$$

(37) If $du = \dfrac{(1+x^2)\,dx}{(1-x^2)(1+x^4)^{\frac{1}{2}}}$, we have by assuming

$$z = \frac{2^{\frac{1}{2}}x}{1-x^2},$$

$$\int\frac{(1+x^2)\,dx}{(1-x^2)(1+x^4)^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}}\int\frac{dz}{(1+z^2)^{\frac{1}{2}}}$$

$$= \frac{1}{2^{\frac{1}{2}}}\log\left\{\frac{(1+x^4)^{\frac{1}{2}}+2^{\frac{1}{2}}x}{1-x^2}\right\}.$$

(38) If $du = \dfrac{(1-x^2)\,dx}{(1+x^2)(1+x^4)^{\frac{1}{2}}}$, we find by assuming

$$z = \frac{2^{\frac{1}{2}}x}{1+x^2},$$

$$\int\frac{dx\,(1-x^2)}{(1+x^2)(1+x^4)^{\frac{1}{2}}} = \frac{1}{2^{\frac{1}{2}}}\sin^{-1}\left(\frac{2^{\frac{1}{2}}x}{1+x^2}\right).$$

(39) If $du = \dfrac{dx\,(1+x^4)^{\frac{1}{2}}}{1-x^4}$, assume $z = \dfrac{2^{\frac{1}{2}}x}{(1+x^4)^{\frac{1}{2}}}$;

therefore $(1+z^2)^{\frac{1}{2}} = \dfrac{1+x^2}{(1+x^4)^{\frac{1}{2}}}$, and $(1-z^2)^{\frac{1}{2}} = \dfrac{1-x^2}{(1+x^4)^{\frac{1}{2}}}$;

and $\dfrac{dx}{(1+x^4)^{\frac{1}{2}}} = \dfrac{1}{2^{\frac{1}{2}}}\cdot\dfrac{dz}{(1-z^4)^{\frac{1}{2}}}$.

Dividing both sides of this equation by

$$\frac{1-x^4}{1+x^4} = (1-z^4)^{\frac{1}{2}},$$

we have

$$\int\frac{dx\,(1+x^4)^{\frac{1}{2}}}{1-x^4} = \frac{1}{2^{\frac{1}{2}}}\int\frac{dz}{1-z^4} = \frac{1}{2^{\frac{3}{2}}}\left(\int\frac{dz}{1+z^2}+\int\frac{dz}{1-z^2}\right)$$

$$= \frac{1}{2^{\frac{3}{2}}}\log\left\{\frac{(1+x^4)^{\frac{1}{2}}+2^{\frac{1}{2}}x}{1-x^2}\right\} + \frac{1}{2^{\frac{3}{2}}}\sin^{-1}\left(\frac{2^{\frac{1}{2}}x}{1+x^2}\right).$$

(40) By the same assumption we find that

$$\int\frac{dx\,x^2}{(1-x^4)(1+x^4)^{\frac{1}{2}}} = \frac{1}{2^{\frac{5}{2}}}\log\left\{\frac{(1+x^4)^{\frac{1}{2}}+2^{\frac{1}{2}}x}{1-x^2}\right\} - \frac{1}{2^{\frac{5}{2}}}\sin^{-1}\left(\frac{2^{\frac{1}{2}}x}{1+x^2}\right).$$

(41) If $du = \dfrac{dx}{(1 - x^2)(2x^2 - 1)^{\frac{1}{4}}}$, assume

$x = z(2x^2 - 1)^{\frac{1}{4}}$, when it becomes

$du = \dfrac{dz}{z^4 - 1}$; and therefore

$$u = \frac{1}{4}\log\left\{\frac{(2x^2 - 1)^{\frac{1}{4}} - x}{(2x^2 - 1)^{\frac{1}{4}} + x}\right\} - \frac{1}{2}\tan^{-1}\frac{x}{(2x^2 - 1)^{\frac{1}{4}}}.$$

(42) If $du = \dfrac{dx}{(1 + x^4)\left\{(1 + x^4)^{\frac{1}{2}} - x^2\right\}^{\frac{1}{2}}}$, we find by assuming $x = z\left\{(1 + x)^{\frac{1}{2}} - x^2\right\}^{\frac{1}{2}}$,

$$u = \tan^{-1}\frac{x}{\left\{(1 + x^4)^{\frac{1}{2}} - x^2\right\}^{\frac{1}{2}}}.$$

(43) In like manner by assuming $x = z\left\{(1 + x^6)^{\frac{1}{3}} - x^2\right\}^{\frac{1}{2}}$, we find

$$\int\frac{dx}{(1 + x^6)\left\{(1 + x^6)^{\frac{1}{3}} - x^2\right\}^{\frac{1}{2}}} = \tan^{-1}\frac{x}{\left\{(1 + x^6)^{\frac{1}{3}} - x^2\right\}^{\frac{1}{2}}}.$$

These transformations are taken from Euler, *Calc. Diff.* Vol. IV. Sup. I.

CHAPTER II.

INTEGRATION BY SUCCESSIVE REDUCTION.

THE method of integration by successive reduction is applicable to a great number of functions, and is the process which in practice is generally the most convenient. I shall here give only the principal formulæ of reduction with a few examples of each, taken chiefly from those integrals which more commonly occur in analysis. The reader who wishes for more numerous examples of the formulæ is referred to the *Integral Tables* compiled by Meyer Hirsch, from which work a great number of the examples in this and the preceding chapter have been taken.

Ex. (1) Let the function to be integrated be

$$\frac{x^n}{(a^2 - x^2)^{\frac{1}{2}}}.$$

The formula of reduction is

$$\int \frac{dx\, x^n}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{x^{n-1}(a^2 - x^2)^{\frac{1}{2}}}{n} + \frac{n-1}{n} a^2 \int \frac{dx\, x^{n-2}}{(a^2 - x^2)^{\frac{1}{2}}}.$$

By this the integral is reduced to

$$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \sin^{-1}\frac{x}{a} \text{ when } n \text{ is even,}$$

$$\text{and to } \int \frac{x\,dx}{(a^2 - x^2)^{\frac{1}{2}}} = -(a^2 - x^2)^{\frac{1}{2}} \text{ when } n \text{ is odd.}$$

Let $n = 3$;

$$\int \frac{x^3 dx}{(a^2 - x^2)^{\frac{1}{2}}} = -\frac{(a^2 - x^2)^{\frac{1}{2}}}{3}(x^2 + 2a^2).$$

Let $n = 6$;

$$\int \frac{x^6 dx}{(a^2 - x^2)^{\frac{1}{2}}} = -(a^2 - x^2)^{\frac{1}{2}}\left(\frac{x^5}{6} + \frac{5a^2x^3}{6.4} + \frac{5.3a^4x}{6.4.2}\right) + \frac{5.3}{6.4.2}a^6 \sin^{-1}\frac{x}{a}.$$

(2) Let the function be

$$\frac{x^n}{(2ax - x^2)^{\frac{1}{2}}}.$$

The formula of reduction is

$$\int \frac{x^n\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -\frac{x^{n-1}(2ax - x^2)^{\frac{1}{2}}}{n} + \frac{2n-1}{n}a\int\frac{x^{n-1}\,dx}{(2ax - x^2)^{\frac{1}{2}}}.$$

By means of this the integral is made to depend on

$$\int \frac{dx}{(2ax - x^2)^{\frac{1}{2}}} = \text{vers}^{-1}\frac{x}{a}.$$

Let $n = 2$;

$$\int \frac{x^2\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -(2ax - x^2)^{\frac{1}{2}}\left(\frac{x}{2} + \frac{3a}{2}\right) + \frac{3a^2}{2}\text{vers}^{-1}\frac{x}{a}.$$

Let $n = 5$;

$$\int \frac{x^5\,dx}{(2ax - x^2)^{\frac{1}{2}}} = -(2ax - x^2)^{\frac{1}{2}}\left(\frac{x^4}{5} + \frac{9}{5.4}ax^3 + \frac{9.7}{5.4.3}a^2x^2\right.$$

$$\left. + \frac{9.7.5}{5.4.3.2}a^3x + \frac{9.7.5.3}{5.4.3.2.1}a^4\right) + \frac{9.7.5.3}{5.4.3.2}\text{vers}^{-1}\frac{x}{a}.$$

(3) Let the function be $\dfrac{1}{(a^2 + x^2)^n}$.

The formula of reduction is

$$\int \frac{dx}{(a^2 + x^2)^n} = \frac{1}{2n-2}\,\frac{x}{a^2(a^2 + x^2)^{n-1}} + \frac{2n-3}{2n-2}\,\frac{1}{a^2}\int\frac{dx}{(a^2 + x^2)^{n-1}}.$$

By this the integral is reduced to

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}.$$

Let $n = 4$;

$$\int \frac{dx}{(a^2 + x^2)^4} = \frac{1}{6}\,\frac{x}{a^2(a^2 + x^2)^3} + \frac{5}{6.4}\,\frac{x}{a^4(a^2 + x^2)^2}$$

$$+ \frac{5.3}{6.4.2}\,\frac{x}{a^6(a^2 + x^2)} + \frac{5.3}{6.4.2}\,\frac{1}{a^7}\tan^{-1}\frac{x}{a}.$$

(4) Let the function be $\dfrac{x^n}{(a^2 + x^2)^m}$.

The formula of reduction is

$$\int\frac{x^n\,dx}{(a^2+x^2)^m}=-\frac{1}{2m-2}\,\frac{x^{n-1}}{(a^2+x^2)^{m-1}}+\frac{n-1}{2m-2}\int\frac{x^{n-2}\,dx}{(a^2+x^2)^{m-1}}.$$

Let $n=2$, $m=3$;

$$\int\frac{x^2\,dx}{(a^2+x^2)^3}=-\frac{x}{4(a^2+x^2)^2}+\frac{x}{4.2a^2(a^2+x^2)}+\frac{1}{4.2}\,\frac{1}{a^3}\tan^{-1}\frac{x}{a}.$$

Let $n=4$, $m=2$;

$$\int\frac{x^4\,dx}{(a^2+x^2)^2}=-\frac{x^3}{2(a^2+x^2)}+\frac{3}{2}\left(x-a\tan^{-1}\frac{x}{a}\right).$$

(5) Let the function be $(a^2-x^2)^{\frac{n}{2}}$, n being odd;

$$\int(a^2-x^2)^{\frac{n}{2}}dx=\frac{x(a^2-x^2)^{\frac{n}{2}}}{n+1}+\frac{na^2}{n+1}\int(a^2-x^2)^{\frac{n-2}{2}}dx.$$

Let $n=1$;

$$\int(a^2-x^2)^{\frac{1}{2}}dx=\frac{x(a^2-x^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Let $n=5$;

$$\int(a^2-x^2)^{\frac{5}{2}}dx=\frac{x(a^2-x^2)^{\frac{5}{2}}}{6}+\frac{5}{6.4}a^2x(a^2-x^2)^{\frac{3}{2}}$$

$$+\frac{5.3}{6.4.2}a^4x(a^2-x^2)^{\frac{1}{2}}+\frac{5.3}{6.4.2}a^6\sin^{-1}\frac{x}{a}.$$

(6) Let the function be $\dfrac{1}{x^n(x^2-1)^{\frac{1}{2}}}$;

$$\int\frac{dx}{x^n(x^2-1)^{\frac{1}{2}}}=\frac{1}{n-1}\,\frac{(x^2-1)^{\frac{1}{2}}}{x^{n-1}}+\frac{n-2}{n-1}\int\frac{dx}{x^{n-2}(x^2-1)^{\frac{1}{2}}}.$$

By this means the integral is reduced to

$$\int\frac{dx}{x(x^2-1)^{\frac{1}{2}}}=\sec^{-1}x \text{ when } n \text{ is odd,}$$

and to $\displaystyle\int\frac{dx}{x^2(x^2-1)^{\frac{1}{2}}}=\frac{(x^2-1)^{\frac{1}{2}}}{x}$ when n is even.

Let $n = 3$;

$$\int \frac{dx}{x^3 (x^2 - 1)^{\frac{1}{2}}} = \frac{(x^2 - 1)^{\frac{1}{2}}}{2x^2} + \tfrac{1}{2} \sec^{-1} x.$$

Let $n = 4$;

$$\int \frac{dx}{x^4 (x^2 - 1)^{\frac{1}{2}}} = \frac{(x^2 - 1)^{\frac{1}{2}}}{3x^3} + \frac{2}{3} \frac{(x^2 - 1)^{\frac{1}{2}}}{x}.$$

(7) If the function be $\dfrac{1}{x^n (1 + x^2)^{\frac{1}{2}}}$ the formula of reduction is the same, excepting that both terms are negative. The final integrals to which it is reduced are

$$\int \frac{dx}{x (1 + x^2)^{\frac{1}{2}}} = \log \left\{ \frac{(1 + x^2)^{\frac{1}{2}} - 1}{x} \right\} \text{ when } n \text{ is odd,}$$

and $\displaystyle\int \frac{dx}{x^2 (1 + x^2)^{\frac{1}{2}}} = - \frac{(1 + x^2)^{\frac{1}{2}}}{x}$ when n is even.

Let $n = 6$;

$$\int \frac{dx}{x^6 (1 + x^2)^{\frac{1}{2}}} = - \frac{1}{5} \frac{(1 + x^2)^{\frac{1}{2}}}{x^5} + \frac{4}{5 . 3} \frac{(1 + x^2)^{\frac{1}{2}}}{x^3} - \frac{4 . 2}{5 . 3} \frac{(1 + x^2)^{\frac{1}{2}}}{x}.$$

(8) Let the function be $\dfrac{x^m}{(a + bx)^{\frac{1}{2}}}$;

$$\int \frac{x^m \, dx}{(a + bx)^{\frac{1}{2}}} = \frac{2x^m (a + bx)^{\frac{1}{2}}}{(2m + 1) b} - \frac{2m}{2m + 1} \frac{a}{b} \int \frac{x^{m-1} \, dx}{(a + bx)^{\frac{1}{2}}}.$$

Let $m = 3$;

$$\int \frac{x^3 dx}{(a + bx)^{\frac{1}{2}}} = \left(\frac{x^3}{7b} - \frac{6}{7.5} \frac{ax^2}{b^2} + \frac{6.4}{7.5.3} \frac{a^2 x}{b^3} - \frac{6.4.2}{7.5.3} \frac{a^3}{b^4} \right) 2(a + bx)^{\frac{1}{2}}.$$

Let $m = 4$, $b = 1$;

$$\int \frac{x^4 \, dx}{(a + x)^{\frac{1}{2}}} = 2 (a + x)^{\frac{1}{2}} \left\{ \frac{x^4}{9} - \frac{8}{9.7} a x^3 + \frac{8.6}{9.7.5} a^2 x^2 - \frac{8.6.4}{9.7.5.3} a^3 x \right.$$

$$\left. + \frac{8.6.4}{9.7.5.3} 2a^4 \right\}.$$

(9) Let the function be $\dfrac{1}{x^n(a+bx)^{\frac{1}{2}}}$;

$$\int\frac{dx}{x^n(a+bx)^{\frac{1}{2}}}=-\frac{1}{(n-1)a}\frac{(a+bx)^{\frac{1}{2}}}{x^{n-1}}-\frac{b}{2a}\frac{2n-3}{n-1}\int\frac{dx}{x^{n-1}(a+bx)^{\frac{1}{2}}}.$$

By means of this the integral is reduced to

$$\int\frac{dx}{x(a+bx)^{\frac{1}{2}}}=\frac{1}{a^{\frac{1}{2}}}\log\frac{(a+bx)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(a+bx)^{\frac{1}{2}}+a^{\frac{1}{2}}}.$$

Let $n=3$;

$$\int\frac{dx}{x^3(a+bx)^{\frac{1}{2}}}=\left(-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right)(a+bx)^{\frac{1}{2}}+\frac{3b^2}{8a^2}\int\frac{dx}{x(a+bx)^{\frac{1}{2}}}.$$

(10) Let the function be $\dfrac{x^n}{(a+bx)^{\frac{1}{3}}}$;

$$\int\frac{x^n\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3x^n(a+bx)^{\frac{2}{3}}}{(3n+2)b}-\frac{3n}{3n+2}\frac{a}{b}\int\frac{x^{n-1}\,dx}{(a+bx)^{\frac{1}{3}}}.$$

If $n=1$;

$$\int\frac{x\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3(a+bx)^{\frac{2}{3}}}{5b^2}\left(bx-\frac{a}{2}\right).$$

If $n=2$;

$$\int\frac{x^2\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3(a+bx)^{\frac{2}{3}}}{b^3}\left\{\frac{(a+bx)^2}{8}-\frac{2}{5}a(a+bx)+\frac{a^2}{2}\right\}.$$

(11) Let the function be $\dfrac{1}{(a^2+x^2)^{\frac{n}{2}}}$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{n}{2}}}=\frac{1}{(n-2)a^2}\frac{x}{(a^2+x^2)^{\frac{n}{2}-1}}+\frac{n-3}{(n-2)a^2}\int\frac{dx}{(a^2+x^2)^{\frac{n}{2}-1}}.$$

If $n=5$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{5}{2}}}=\left(\frac{1}{a^2+x^2}+\frac{2}{a^2}\right)\frac{x}{3a^2(a^2+x^2)^{\frac{1}{2}}}.$$

If $n=7$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{7}{2}}}=\left\{\frac{1}{(a^2+x^2)^2}+\frac{4}{3a^2(a^2+x^2)}+\frac{1}{3a^4}\right\}\frac{x}{5a^2(a^2+x^2)^{\frac{1}{2}}}.$$

(12) Let the function be $\dfrac{x^n}{(a+bx+cx^2)^{\frac{1}{2}}}$;

$$\int\frac{x^n\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}=\frac{x^{n-1}(a+bx+cx^2)^{\frac{1}{2}}}{nc}-\frac{n-1}{n}\frac{a}{c}\int\frac{x^{n-2}}{(a+bx+cx^2)^{\frac{1}{2}}}$$
$$-\frac{2n-1}{2n}\frac{b}{c}\int\frac{x^{n-1}\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}.$$

By this the integral is made to depend on

$$\int\frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}},\ \text{and}\ \int\frac{x\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}.$$

If $n=2$, $a=b=c=1$;

$$\int\frac{x^2\,dx}{(1+x+x^2)^{\frac{1}{2}}}=\frac{2x-3}{4}(1+x+x^2)^{\frac{1}{2}}-\frac{1}{8}\log\{2x+1+2(1+x+x^2)^{\frac{1}{2}}\}.$$

If $n=3$, $a=1$, $b=c=-1$;

$$\int\frac{x^3\,dx}{(1-x-x^2)^{\frac{1}{2}}}=-\frac{(1-x-x^2)^{\frac{1}{2}}}{24}(8x^2-10x+31)-\frac{17}{16}\sin^{-1}\frac{2x+1}{5^{\frac{1}{2}}}.$$

(13) Let the function be $\epsilon^{ax}x^n$;

$$\int\epsilon^{ax}x^n\,dx=\frac{\epsilon^{ax}x^n}{a}-\frac{n}{a}\int\epsilon^{ax}x^{n-1}\,dx.$$

If $n=4$;

$$\int\epsilon^{ax}x^4\,dx=\epsilon^{ax}\left\{\frac{x^4}{a}-\frac{4x^3}{a^2}+\frac{4.3.x^2}{a^3}-\frac{4.3.2x}{a^4}+\frac{4.3.2.1}{a^5}\right\}.$$

If $n=5$, $a=-1$;

$$\int\epsilon^{-x}x^5\,dx=-\epsilon^{-x}(x^5+5x^3+20x^3+60x^2+120x+120).$$

(14) Let the function be $\dfrac{\epsilon^{ax}}{x^n}$.

The formula of reduction is

$$\int\frac{\epsilon^{ax}\,dx}{x^n}=-\frac{\epsilon^{ax}}{(n-1)x^{n-1}}+\frac{a}{n-1}\int\frac{\epsilon^{ax}}{x^{n-1}}dx,$$

by means of which the integral is reduced to

$$\int\frac{\epsilon^{ax}\,dx}{x}=\log x+ax+\frac{a^2x^2}{1.2^2}+\frac{a^3x^3}{1.2.3^2}+\&c.$$

If $n = 3, \quad a = 1$;

$$\int \frac{\epsilon^x\,dx}{x^3} = -\frac{\epsilon^x}{2x^2}(1 + x) + \tfrac{1}{2}\int \frac{\epsilon^x\,dx}{x}.$$

(15) Let the function be $x^m (\log x)^n$;

$$\int x^m (\log x)^n\,dx = \frac{x^{m+1}(\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\,dx.$$

If $m = 3, \quad n = 2$;

$$\int x^3 (\log x)^2\,dx = \frac{x^4}{4}\left\{(\log x)^2 - \tfrac{1}{2}\log x + \tfrac{1}{8}\right\}.$$

If $m = 1, \quad n = 3$;

$$\int x (\log x)^3\,dx = \frac{x^2}{2}\left\{(\log x)^3 - \frac{3}{2}(\log x)^2 + \frac{3}{2}(\log x) - \frac{3}{2^2}\right\}.$$

(16) Let the function be $(\sin x)^m (\cos x)^n$.

We may use any of the following formulæ of reduction:

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+1} + \frac{n-1}{m+1}\int dx\,(\sin x)^{m+2}(\cos x)^{n-2} \ldots (1)$$

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m-1}(\cos x)^{n+1}}{n+1} + \frac{m-1}{n+1}\int dx\,(\sin x)^{m-2}(\cos x)^{n+2} \ldots (2)$$

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m-1}(\cos x)^{n+1}}{m+n} + \frac{m-1}{m+n}\int dx\,(\sin x)^{m-2}(\cos x)^n \ldots (3)$$

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n}\int dx\,(\sin x)^m (\cos x)^{n-2} \ldots (4)$$

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n+1}}{m+1} + \frac{m+n+2}{m+1}\int dx\,(\sin x)^{m+2}(\cos x)^n \ldots (5)$$

$$\int dx\,(\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m+1}(\cos x)^{n+1}}{n+1} + \frac{m+n+2}{n+1}\int dx\,(\sin x)^m (\cos x)^{n+2} \ldots (6)$$

$$\int dx\,(\sin x)^3 = -\frac{\cos x}{3}\left\{(\sin x)^2 + 2\right\} \text{ by (3)}$$

$$= \tfrac{1}{4}\left(\frac{\cos 3x}{3} - 3\cos x\right).$$

$$\int dx\,(\sin x)^4 = -\frac{\cos x}{4}\left\{(\sin x)^3 + \frac{3}{2}\sin x\right\} + \frac{3x}{8} \text{ by (3)}$$

$$= \tfrac{1}{4}\left(\frac{\sin 4x}{8} - \sin 2x\right) + \frac{3x}{8}.$$

$$\int dx\,(\cos x)^2 = \tfrac{1}{2}(\sin x\cos x + x) \text{ by (4)}$$

$$= \tfrac{1}{2}\left(x + \frac{\sin 2x}{2}\right).$$

$$\int dx\,(\cos x)^5 = \frac{\sin x}{5}\left\{(\cos x)^4 + \frac{4}{3}(\cos x)^2 + \frac{8}{3}\right\} \text{ by (4)}$$

$$= \tfrac{1}{8}\left\{\frac{\sin 5x}{10} + \frac{5\sin 3x}{6} + 5\sin x\right\}.$$

$$\int dx\,(\sin x)^2(\cos x)^3 = \frac{(\sin x)^3}{5}\left\{(\cos x)^2 + \frac{2}{3}\right\} \text{ by (4),}$$

$$= -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right).$$

$$\int dx\,(\sin x)^4(\cos x)^3 = \frac{(\sin x)^5}{7}\left\{(\cos x)^2 + \frac{2}{5}\right\}.$$

$$\int dx\,(\sin x)^5(\cos x)^5 = -\frac{1}{2^9}\left(\frac{1}{10}\cos 10x - \frac{5}{6}\cos 6x + 5\cos 2x\right).$$

$$\int dx\,(\sin x)^7(\cos x)^2$$

$$= \frac{1}{2^8}\left(\frac{1}{9}\cos 9x - \frac{5}{7}\cos 7x + \frac{8}{5}\cos 5x - 14\cos x\right).$$

$$\int \frac{dx}{(\sin x)^6} = -\frac{\cos x}{5}\left\{\frac{1}{(\sin x)^5} + \frac{4}{3}\,\frac{1}{(\sin x)^3} + \frac{8}{3}\,\frac{1}{\sin x}\right\} \text{ by (5).}$$

$$\int \frac{dx}{(\sin x)^3} = -\frac{\cos x}{2(\sin x)^2} + \frac{1}{2}\log\left(\tan\frac{x}{2}\right).$$

$$\int \frac{dx}{(\cos x)^4} = \frac{\sin x}{3}\left\{\frac{1}{(\cos x)^3} + \frac{2}{\cos x}\right\} \text{ by (6)},$$

$$= \tan x + \tfrac{1}{3}(\tan x)^3.$$

$$\int \frac{dx}{(\cos x)^5} = \frac{\sin x}{4}\left\{\frac{1}{(\cos x)^4} + \frac{3}{2(\cos x)^2}\right\} + \frac{3}{8}\log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

$$\int \frac{dx\,(\sin x)^5}{\cos x} = -\tfrac{1}{2}\left\{\frac{(\sin x)^4}{2} + (\sin x)^2\right\} + \log(\sec x) \text{ by (3)}.$$

$$\int \frac{dx\,(\cos x)^4}{\sin x} = \frac{(\cos x)^3}{3} + \cos x + \log\left(\tan\frac{x}{2}\right) \text{ by (4)}.$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^2} = \cos x + \sec x.$$

$$\int \frac{dx\,(\cos x)^4}{(\sin x)^2} = \frac{1}{2\sin x}\left\{(\cos x)^3 - 3\cos x\right\} - \frac{3}{2}x.$$

$$\int \frac{dx\,(\cos x)^5}{(\sin x)^3} = \frac{1}{(\sin x)^2}\left\{\frac{(\cos x)^4}{2} - 1\right\} - 2\log(\sin x).$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^1} = \left\{(\sin x)^2 - \frac{2}{3}\right\}\frac{1}{(\cos x)^3}.$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^8} = \frac{1}{5(\cos x)^7}\left\{\sin x - \frac{2}{7}\right\}.$$

$$\int \frac{dx}{\sin x\,(\cos x)^3} = \frac{1}{2(\cos x)^2} + \log(\tan x).$$

$$\int \frac{dx}{(\sin x)^2(\cos x)^4} = \frac{1}{3\sin x\,(\cos x)^3} - \frac{8}{3}\cot 2x.$$

$$\int \frac{dx}{(\sin x)^1(\cos x)^4} = -\frac{8\cos 2x}{3}\left\{\frac{1}{(\sin 2x)^3} + \frac{2}{\sin 2x}\right\}.$$

$$\int \frac{dx}{(\sin x)^5\cos x} = -\frac{1}{4(\sin x)^4} - \frac{1}{2(\sin x)^2} + \log(\tan x).$$

(17) If the function be $(\tan x)^n$ the formula of reduction is

$$\int dx\,(\tan x)^n = \frac{(\tan x)^{n-1}}{n-1} - \int dx\,(\tan x)^{n-2}.$$

$$\int dx\,(\sin x)^3 = -\frac{\cos x}{3}\left\{(\sin x)^2 + 2\right\} \text{ by } (3)$$

$$= \tfrac{1}{4}\left(\frac{\cos 3x}{3} - 3\cos x\right).$$

$$\int dx\,(\sin x)^4 = -\frac{\cos x}{4}\left\{(\sin x)^3 + \frac{3}{2}\sin x\right\} + \frac{3x}{8} \text{ by } (3)$$

$$= \tfrac{1}{4}\left(\frac{\sin 4x}{8} - \sin 2x\right) + \frac{3x}{8}.$$

$$\int dx\,(\cos x)^2 = \tfrac{1}{2}(\sin x\cos x + x) \text{ by } (4)$$

$$= \tfrac{1}{2}\left(x + \frac{\sin 2x}{2}\right).$$

$$\int dx\,(\cos x)^5 = \frac{\sin x}{5}\left\{(\cos x)^4 + \frac{4}{3}(\cos x)^2 + \frac{8}{3}\right\} \text{ by } (4)$$

$$= \tfrac{1}{8}\left\{\frac{\sin 5x}{10} + \frac{5\sin 3x}{6} + 5\sin x\right\}.$$

$$\int dx\,(\sin x)^2(\cos x)^3 = \frac{(\sin x)^3}{5}\left\{(\cos x)^2 + \frac{2}{3}\right\} \text{ by } (4),$$

$$= -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right).$$

$$\int dx\,(\sin x)^4(\cos x)^3 = \frac{(\sin x)^5}{7}\left\{(\cos x)^2 + \frac{2}{5}\right\}.$$

$$\int dx\,(\sin x)^5(\cos x)^5 = -\frac{1}{2^9}\left(\frac{1}{10}\cos 10x - \frac{5}{6}\cos 6x + 5\cos 2x\right).$$

$$\int dx\,(\sin x)^7(\cos x)^2$$

$$= \frac{1}{2^8}\left(\frac{1}{9}\cos 9x - \frac{5}{7}\cos 7x + \frac{8}{5}\cos 5x - 14\cos x\right).$$

$$\int \frac{dx}{(\sin x)^6} = -\frac{\cos x}{5}\left\{\frac{1}{(\sin x)^5} + \frac{4}{3}\,\frac{1}{(\sin x)^3} + \frac{8}{3}\,\frac{1}{\sin x}\right\} \text{ by } (5).$$

$$\int \frac{dx}{(\sin x)^3} = -\frac{\cos x}{2(\sin x)^2} + \frac{1}{2}\log\left(\tan\frac{x}{2}\right).$$

$$\int dx\,(\sin x)^3 = -\frac{\cos x}{3}\left\{(\sin x)^2 + 2\right\} \text{ by (3)}$$

$$= \frac{1}{4}\left(\frac{\cos 3x}{3} - 3\cos x\right).$$

$$\int dx\,(\sin x)^4 = -\frac{\cos x}{4}\left\{(\sin x)^3 + \frac{3}{2}\sin x\right\} + \frac{3x}{8} \text{ by (3)}$$

$$= \frac{1}{4}\left(\frac{\sin 4x}{8} - \sin 2x\right) + \frac{3x}{8}.$$

$$\int dx\,(\cos x)^2 = \frac{1}{2}\left(\sin x\cos x + x\right) \text{ by (4)}$$

$$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right).$$

$$\int dx\,(\cos x)^5 = \frac{\sin x}{5}\left\{(\cos x)^4 + \frac{4}{3}(\cos x)^2 + \frac{8}{3}\right\} \text{ by (4)}$$

$$= \frac{1}{8}\left\{\frac{\sin 5x}{10} + \frac{5\sin 3x}{6} + 5\sin x\right\}.$$

$$\int dx\,(\sin x)^2(\cos x)^3 = \frac{(\sin x)^3}{5}\left\{(\cos x)^2 + \frac{2}{3}\right\} \text{ by (4),}$$

$$= -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right).$$

$$\int dx\,(\sin x)^4(\cos x)^3 = \frac{(\sin x)^5}{7}\left\{(\cos x)^2 + \frac{2}{5}\right\}.$$

$$\int dx\,(\sin x)^5(\cos x)^5 = -\frac{1}{2^9}\left(\frac{1}{10}\cos 10x - \frac{5}{6}\cos 6x + 5\cos 2x\right).$$

$$\int dx\,(\sin x)^7(\cos x)^2$$

$$= \frac{1}{2^8}\left(\frac{1}{9}\cos 9x - \frac{5}{7}\cos 7x + \frac{8}{5}\cos 5x - 14\cos x\right).$$

$$\int \frac{dx}{(\sin x)^6} = -\frac{\cos x}{5}\left\{\frac{1}{(\sin x)^5} + \frac{4}{3}\,\frac{1}{(\sin x)^3} + \frac{8}{3}\,\frac{1}{\sin x}\right\} \text{ by (5).}$$

$$\int \frac{dx}{(\sin x)^3} = -\frac{\cos x}{2(\sin x)^2} + \frac{1}{2}\log\left(\tan\frac{x}{2}\right).$$

$$\int\frac{dx}{(\cos x)^4}=\frac{\sin x}{3}\left\{\frac{1}{(\cos x)^3}+\frac{2}{\cos x}\right\} \text{ by (6),}$$

$$= \tan x+\tfrac{1}{3}(\tan x)^3.$$

$$\int\frac{dx}{(\cos x)^5}=\frac{\sin x}{4}\left\{\frac{1}{(\cos x)^4}+\frac{3}{2(\cos x)^2}\right\}+\frac{3}{8}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right).$$

$$\int\frac{dx\,(\sin x)^5}{\cos x}=-\tfrac{1}{2}\left\{\frac{(\sin x)^4}{2}+(\sin x)^2\right\}+\log(\sec x) \text{ by (3).}$$

$$\int\frac{dx\,(\cos x)^4}{\sin x}=\frac{(\cos x)^3}{3}+\cos x+\log\left(\tan\frac{x}{2}\right) \text{ by (4).}$$

$$\int\frac{dx\,(\sin x)^3}{(\cos x)^2}=\cos x+\sec x.$$

$$\int\frac{dx\,(\cos x)^4}{(\sin x)^2}=\frac{1}{2\sin x}\left\{(\cos x)^3-3\cos x\right\}-\frac{3}{2}x.$$

$$\int\frac{dx\,(\cos x)^5}{(\sin x)^3}=\frac{1}{(\sin x)^2}\left\{\frac{(\cos x)^4}{2}-1\right\}-2\log(\sin x).$$

$$\int\frac{dx\,(\sin x)^3}{(\cos x)^4}=\left\{(\sin x)^2-\frac{2}{3}\right\}\frac{1}{(\cos x)^3}.$$

$$\int\frac{dx\,(\sin x)^3}{(\cos x)^8}=\frac{1}{5(\cos x)^7}\left\{\sin x-\frac{2}{7}\right\}.$$

$$\int\frac{dx}{\sin x\,(\cos x)^3}=\frac{1}{2(\cos x)^2}+\log(\tan x).$$

$$\int\frac{dx}{(\sin x)^2(\cos x)^4}=\frac{1}{3\sin x\,(\cos x)^3}-\frac{8}{3}\cot 2x.$$

$$\int\frac{dx}{(\sin x)^4(\cos x)^4}=-\frac{8\cos 2x}{3}\left\{\frac{1}{(\sin 2x)^3}+\frac{2}{\sin 2x}\right\}.$$

$$\int\frac{dx}{(\sin x)^5\cos x}=-\frac{1}{4(\sin x)^4}-\frac{1}{2(\sin x)^2}+\log(\tan x).$$

(17) If the function be $(\tan x)^n$ the formula of reduction is

$$\int dx\,(\tan x)^n=\frac{(\tan x)^{n-1}}{n-1}-\int dx\,(\tan x)^{n-2}.$$

If the function be $\frac{1}{(\tan x)^n}$ the formula of reduction is

$$\int \frac{dx}{(\tan x)^n} = -\frac{1}{(m-1)}\frac{1}{(\tan x)^{n-1}} - \int dx \frac{1}{(\tan x)^{n-2}}.$$

$$\int dx\, (\tan x)^4 = \frac{1}{3}(\tan x)^3 - \tan x + x.$$

$$\int dx\, (\tan x)^7 = \frac{1}{6}(\tan x)^6 - \frac{1}{4}(\tan x)^4 + \frac{1}{2}(\tan x)^2 + \log(\cos x).$$

$$\int \frac{dx}{(\tan x)^5} = -\tfrac{1}{4}(\cot x)^4 + \tfrac{1}{2}(\cot x)^2 + \log(\sin x).$$

(18) If the function be $x^n \cos x$, the formula of reduction is

$$\int dx\, x^n \cos x = x^n \sin x + n x^{n-1} \cos x - n(n-1)\int dx\, x^{n-2} \cos x.$$

$$\int dx\, x^2 \cos x = x^2 \sin x + 2x \cos x - 2 \sin x.$$

$$\int dx\, x^3 \cos x = x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6 \cos x.$$

In the same way we find

$$\int dx\, x \sin x = - x \cos x + \sin x.$$

$$\int dx\, x^4 \sin x = - x^4 \cos x + 4x^3 \sin x + 12x^2 \cos x - 24x \sin x - 24 \cos x.$$

(19) If the function be $\epsilon^{ax}(\cos x)^n$ the formula of reduction is

$$\int dx\, \epsilon^{ax}(\cos x)^n$$
$$= \frac{\epsilon^{ax}(\cos x)^{n-1}(a \cos x + n \sin x)}{a^2 + n^2} + \frac{n(n-1)}{a^2+n^2}\int dx\, \epsilon^{ax}(\cos x)^{n-2};$$

a similar formula exists for $\epsilon^{ax}(\sin x)^n$.

$$\int dx\, \epsilon^{ax}(\cos x)^2 = \epsilon^{ax} \cos x \frac{(a \cos x + 2 \sin x)}{a^2 + 4} + \frac{2\epsilon^{ax}}{a(a^2+4)}.$$

$$\int dx\, \epsilon^{ax}(\sin x)^3 = \epsilon^{ax}(\sin x)^2 \frac{(a \sin x - 3 \cos x)}{a^2 + 9} + \frac{6\epsilon^{ax}(a \sin x - \cos x)}{(a^2+1)(a^2+9)}.$$

$$\int dx\, \epsilon^{ax}(\sin x)^5 (\cos x)^2 = \frac{\epsilon^{ax}}{64}\left\{\frac{a \sin 7x - 7 \cos 7x}{a^2 + 49}\right.$$
$$\left. - \frac{3(a \sin 5x - 5 \cos 5x)}{a^2 + 25} + \frac{a \sin 3x - 3 \cos 3x}{a^2 + 9} + \frac{5(a \sin x - \cos x)}{a^2 + 1}\right\}.$$

(20) If the function be $\frac{1}{(a+b\cos x)^n}$ the formula of reduction is

$$\int\frac{dx}{(a+b\cos x)^n}=\frac{-b\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-1}}$$

$$+\frac{(2n-3)a}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\cos x)^{n-1}}-\frac{(n-2)}{(n-1)(a^2-b^2)}\int\frac{dx}{(a+b\cos x)^{n-2}}.$$

Let $n=2$, then

$$\int\frac{dx}{(a+b\cos x)^2}$$

$$=\frac{1}{a^2-b^2}\left[\frac{-b\sin x}{a+b\cos x}+\frac{2a}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right\}\right].$$

Hence also we find

$$\int\frac{dx\cos x}{(a+b\cos x)^2}$$

$$=\frac{1}{a^2-b^2}\left[\frac{a\sin x}{a+b\cos x}-\frac{2b}{(a^2-b^2)^{\frac{1}{2}}}\tan^{-1}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\frac{x}{2}\right\}\right].$$

Let $n = 3$;

$$\int \frac{dx}{x^3 (x^2 - 1)^{\frac{1}{2}}} = \frac{(x^2 - 1)^{\frac{1}{2}}}{2x^2} + \frac{1}{2} \sec^{-1} x.$$

Let $n = 4$;

$$\int \frac{dx}{x^4 (x^2 - 1)^{\frac{1}{2}}} = \frac{(x^2 - 1)^{\frac{1}{2}}}{3x^3} + \frac{2}{3} \frac{(x^2 - 1)^{\frac{1}{2}}}{x}.$$

(7) If the function be $\dfrac{1}{x^n (1 + x^2)^{\frac{1}{2}}}$ the formula of reduction is the same, excepting that both terms are negative. The final integrals to which it is reduced are

$$\int \frac{dx}{x (1 + x^2)^{\frac{1}{2}}} = \log \left\{ \frac{(1 + x^2)^{\frac{1}{2}} - 1}{x} \right\} \text{ when } n \text{ is odd,}$$

and $$\int \frac{dx}{x^2 (1 + x^2)^{\frac{1}{2}}} = - \frac{(1 + x^2)^{\frac{1}{2}}}{x} \text{ when } n \text{ is even.}$$

Let $n = 6$;

$$\int \frac{dx}{x^6 (1 + x^2)^{\frac{1}{2}}} = - \frac{1}{5} \frac{(1 + x^2)^{\frac{1}{2}}}{x^5} + \frac{4}{5 . 3} \frac{(1 + x^2)^{\frac{1}{2}}}{x^3} - \frac{4 . 2}{5 . 3} \frac{(1 + x^2)^{\frac{1}{2}}}{x}.$$

(8) Let the function be $\dfrac{x^m}{(a + bx)^{\frac{1}{2}}}$;

$$\int \frac{x^m \, dx}{(a + bx)^{\frac{1}{2}}} = \frac{2x^m (a + bx)^{\frac{1}{2}}}{(2m + 1) b} - \frac{2m}{2m + 1} \frac{a}{b} \int \frac{x^{m-1} \, dx}{(a + bx)^{\frac{1}{2}}}.$$

Let $m = 3$;

$$\int \frac{x^3 dx}{(a + bx)^{\frac{1}{2}}} = \left(\frac{x^3}{7b} - \frac{6}{7.5} \frac{a x^2}{b^2} + \frac{6.4}{7.5.3} \frac{a^2 x}{b^3} - \frac{6.4.2}{7.5.3} \frac{a^2}{b^4} \right) 2(a+bx)^{\frac{1}{2}}.$$

Let $m = 4$, $b = 1$;

$$\int \frac{x^4 \, dx}{(a + x)^{\frac{1}{2}}} = 2 (a + x)^{\frac{1}{2}} \left\{ \frac{x^4}{9} - \frac{8}{9.7} a x^3 + \frac{8.6}{9.7.5} a^2 x^2 - \frac{8.6.4}{9.7.5.3} a^3 x \right.$$

$$\left. + \frac{8.6.4}{9.7.5.3} 2a^4 \right\}.$$

(9) Let the function be $\dfrac{1}{x^n(a+bx)^{\frac{1}{2}}}$;

$$\int\frac{dx}{x^n(a+bx)^{\frac{1}{2}}}=-\frac{1}{(n-1)a}\frac{(a+bx)^{\frac{1}{2}}}{x^{n-1}}-\frac{b}{2a}\frac{2n-3}{n-1}\int\frac{dx}{x^{n-1}(a+bx)^{\frac{1}{2}}}.$$

By means of this the integral is reduced to

$$\int\frac{dx}{x(a+bx)^{\frac{1}{2}}}=\frac{1}{a^{\frac{1}{2}}}\log\frac{(a+bx)^{\frac{1}{2}}-a^{\frac{1}{2}}}{(a+bx)^{\frac{1}{2}}+a^{\frac{1}{2}}}.$$

Let $n=3$;

$$\int\frac{dx}{x^3(a+bx)^{\frac{1}{2}}}=\left(-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right)(a+bx)^{\frac{1}{2}}+\frac{3b^2}{8a^2}\int\frac{dx}{x(a+bx)^{\frac{1}{2}}}.$$

(10) Let the function be $\dfrac{x^n}{(a+bx)^{\frac{1}{3}}}$;

$$\int\frac{x^n\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3x^n(a+bx)^{\frac{2}{3}}}{(3n+2)b}-\frac{3n}{3n+2}\frac{a}{b}\int\frac{x^{n-1}\,dx}{(a+bx)^{\frac{1}{3}}}.$$

If $n=1$;

$$\int\frac{x\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3(a+bx)^{\frac{2}{3}}}{5b^2}\left(bx-\frac{a}{2}\right).$$

If $n=2$;

$$\int\frac{x^2\,dx}{(a+bx)^{\frac{1}{3}}}=\frac{3(a+bx)^{\frac{2}{3}}}{b^3}\left\{\frac{(a+bx)^2}{8}-\frac{2}{5}a(a+bx)+\frac{a^2}{2}\right\}.$$

(11) Let the function be $\dfrac{1}{(a^2+x^2)^{\frac{n}{2}}}$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{n}{2}}}=\frac{1}{(n-2)a^2}\frac{x}{(a^2+x^2)^{\frac{n}{2}-1}}+\frac{n-3}{(n-2)a^2}\int\frac{dx}{(a^2+x^2)^{\frac{n}{2}-1}}.$$

If $n=5$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{5}{2}}}=\left(\frac{1}{a^2+x^2}+\frac{2}{a^2}\right)\frac{x}{3a^2(a^2+x^2)^{\frac{1}{2}}}.$$

If $n=7$;

$$\int\frac{dx}{(a^2+x^2)^{\frac{7}{2}}}=\left\{\frac{1}{(a^2+x^2)^2}+\frac{4}{3a^2(a^2+x^2)}+\frac{1}{3a^4}\right\}\frac{x}{5a^2(a^2+x^2)^{\frac{1}{2}}}.$$

(12) Let the function be $\dfrac{x^n}{(a+bx+cx^2)^{\frac{1}{2}}}$;

$$\int \frac{x^n\,dx}{(a+bx+cx^2)^{\frac{1}{2}}} = \frac{x^{n-1}(a+bx+cx^2)^{\frac{1}{2}}}{nc} - \frac{n-1}{n}\frac{a}{c}\int \frac{x^{n-2}}{(a+bx+cx^2)^{\frac{1}{2}}}$$

$$- \frac{2n-1}{2n}\frac{b}{c}\int \frac{x^{n-1}\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}.$$

By this the integral is made to depend on

$$\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}, \quad \text{and} \quad \int \frac{x\,dx}{(a+bx+cx^2)^{\frac{1}{2}}}.$$

If $n = 2$, $a = b = c = 1$;

$$\int \frac{x^2\,dx}{(1+x+x^2)^{\frac{1}{2}}} = \frac{2x-3}{4}(1+x+x^2)^{\frac{1}{2}} - \frac{1}{8}\log\left\{2x+1+2(1+x+x^2)^{\frac{1}{2}}\right\}.$$

If $n = 3$, $a = 1$, $b = c = -1$;

$$\int \frac{x^3\,dx}{(1-x-x^2)^{\frac{1}{2}}} = -\frac{(1-x-x^2)^{\frac{1}{2}}}{24}(8x^2-10x+31) - \frac{17}{16}\sin^{-1}\frac{2x+1}{5^{\frac{1}{2}}}.$$

(13) Let the function be $\epsilon^{ax}x^n$;

$$\int \epsilon^{ax}x^n\,dx = \frac{\epsilon^{ax}x^n}{a} - \frac{n}{a}\int \epsilon^{ax}x^{n-1}\,dx.$$

If $n = 4$;

$$\int \epsilon^{ax}x^4\,dx = \epsilon^{ax}\left\{\frac{x^4}{a} - \frac{4x^3}{a^2} + \frac{4.3.x^2}{a^3} - \frac{4.3.2x}{a^4} + \frac{4.3.2.1}{a^5}\right\}.$$

If $n = 5$, $a = -1$;

$$\int \epsilon^{-x}x^5\,dx = -\epsilon^{-x}(x^5 + 5x^3 + 20x^3 + 60x^2 + 120x + 120).$$

(14) Let the function be $\dfrac{\epsilon^{ax}}{x^n}$.

The formula of reduction is

$$\int \frac{\epsilon^{ax}\,dx}{x^n} = -\frac{\epsilon^{ax}}{(n-1)\,x^{n-1}} + \frac{a}{n-1}\int \frac{\epsilon^{ax}}{x^{n-1}}\,dx,$$

by means of which the integral is reduced to

$$\int \frac{\epsilon^{ax}\,dx}{x} = \log x + ax + \frac{a^2x^2}{1.2^2} + \frac{a^3x^3}{1.2.3^2} + \&c.$$

If $n = 3, \quad a = 1$;

$$\int \frac{\epsilon^x\, dx}{x^3} = -\frac{\epsilon^x}{2x^2}(1 + x) + \tfrac{1}{2}\int \frac{\epsilon^x\, dx}{x}.$$

(15) Let the function be $x^m (\log x)^n$;

$$\int x^m (\log x)^n\, dx = \frac{x^{m+1} (\log x)^n}{m+1} - \frac{n}{m+1}\int x^m (\log x)^{n-1}\, dx.$$

If $m = 3, \quad n = 2$;

$$\int x^3 (\log x)^2\, dx = \frac{x^4}{4}\left\{(\log x)^2 - \tfrac{1}{2}\log x + \tfrac{1}{8}\right\}.$$

If $m = 1, \quad n = 3$;

$$\int x (\log x)^3\, dx = \frac{x^2}{2}\left\{(\log x)^3 - \frac{3}{2}(\log x)^2 + \frac{3}{2}(\log x) - \frac{3}{2^2}\right\}.$$

(16) Let the function be $(\sin x)^m (\cos x)^n$.

We may use any of the following formulæ of reduction:

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+1} + \frac{n-1}{m+1}\int dx\, (\sin x)^{m+2}(\cos x)^{n-2} \ldots (1)$$

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m-1}(\cos x)^{n+1}}{n+1} + \frac{m-1}{n+1}\int dx (\sin x)^{m-2}(\cos x)^{n+2} \ldots (2)$$

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m-1}(\cos x)^{n+1}}{m+n} + \frac{m-1}{m+n}\int dx\, (\sin x)^{m-2}(\cos x)^n \ldots (3)$$

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n-1}}{m+n} + \frac{n-1}{m+n}\int dx\, (\sin x)^m (\cos x)^{n-2} \ldots (4)$$

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= \frac{(\sin x)^{m+1}(\cos x)^{n+1}}{m+1} + \frac{m+n+2}{m+1}\int dx (\sin x)^{m+2}(\cos x)^n \ldots (5)$$

$$\int dx\, (\sin x)^m (\cos x)^n$$
$$= -\frac{(\sin x)^{m+1}(\cos x)^{n+1}}{n+1} + \frac{m+n+2}{n+1}\int dx (\sin x)^m (\cos x)^{n+2} \ldots (6)$$

$$\int dx\,(\sin x)^3 = -\frac{\cos x}{3}\left\{(\sin x)^2 + 2\right\} \text{ by (3)}$$

$$= \frac{1}{4}\left(\frac{\cos 3x}{3} - 3\cos x\right).$$

$$\int dx\,(\sin x)^4 = -\frac{\cos x}{4}\left\{(\sin x)^3 + \frac{3}{2}\sin x\right\} + \frac{3x}{8} \text{ by (3)}$$

$$= \frac{1}{4}\left(\frac{\sin 4x}{8} - \sin 2x\right) + \frac{3x}{8}.$$

$$\int dx\,(\cos x)^2 = \frac{1}{2}(\sin x \cos x + x) \text{ by (4)}$$

$$= \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right).$$

$$\int dx\,(\cos x)^5 = \frac{\sin x}{5}\left\{(\cos x)^4 + \frac{4}{3}(\cos x)^2 + \frac{8}{3}\right\} \text{ by (4)}$$

$$= \frac{1}{8}\left\{\frac{\sin 5x}{10} + \frac{5\sin 3x}{6} + 5\sin x\right\}.$$

$$\int dx\,(\sin x)^2(\cos x)^3 = \frac{(\sin x)^3}{5}\left\{(\cos x)^2 + \frac{2}{3}\right\} \text{ by (4),}$$

$$= -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right).$$

$$\int dx\,(\sin x)^4(\cos x)^3 = \frac{(\sin x)^5}{7}\left\{(\cos x)^2 + \frac{2}{5}\right\}.$$

$$\int dx\,(\sin x)^5(\cos x)^5 = -\frac{1}{2^9}\left(\frac{1}{10}\cos 10x - \frac{5}{6}\cos 6x + 5\cos 2x\right).$$

$$\int dx\,(\sin x)^7(\cos x)^2$$

$$= \frac{1}{2^8}\left(\frac{1}{9}\cos 9x - \frac{5}{7}\cos 7x + \frac{8}{5}\cos 5x - 14\cos x\right).$$

$$\int \frac{dx}{(\sin x)^6} = -\frac{\cos x}{5}\left\{\frac{1}{(\sin x)^5} + \frac{4}{3}\,\frac{1}{(\sin x)^3} + \frac{8}{3}\,\frac{1}{\sin x}\right\} \text{ by (5).}$$

$$\int \frac{dx}{(\sin x)^3} = -\frac{\cos x}{2(\sin x)^2} + \frac{1}{2}\log\left(\tan\frac{x}{2}\right).$$

$$\int \frac{dx}{(\cos x)^4} = \frac{\sin x}{3}\left\{\frac{1}{(\cos x)^3} + \frac{2}{\cos x}\right\} \text{ by (6)},$$

$$= \tan x + \tfrac{1}{3}(\tan x)^3.$$

$$\int \frac{dx}{(\cos x)^5} = \frac{\sin x}{4}\left\{\frac{1}{(\cos x)^4} + \frac{3}{2(\cos x)^2}\right\} + \frac{3}{8}\log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

$$\int \frac{dx\,(\sin x)^5}{\cos x} = -\tfrac{1}{2}\left\{\frac{(\sin x)^4}{2} + (\sin x)^2\right\} + \log(\sec x) \text{ by (3)}.$$

$$\int \frac{dx\,(\cos x)^4}{\sin x} = \frac{(\cos x)^3}{3} + \cos x + \log\left(\tan\frac{x}{2}\right) \text{ by (4)}.$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^2} = \cos x + \sec x.$$

$$\int \frac{dx\,(\cos x)^4}{(\sin x)^2} = \frac{1}{2\sin x}\{(\cos x)^3 - 3\cos x\} - \frac{3}{2}x.$$

$$\int \frac{dx\,(\cos x)^5}{(\sin x)^3} = \frac{1}{(\sin x)^2}\left\{\frac{(\cos x)^4}{2} - 1\right\} - 2\log(\sin x).$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^4} = \left\{(\sin x)^2 - \frac{2}{3}\right\}\frac{1}{(\cos x)^3}.$$

$$\int \frac{dx\,(\sin x)^3}{(\cos x)^8} = \frac{1}{5(\cos x)^7}\left\{\sin x - \frac{2}{7}\right\}.$$

$$\int \frac{dx}{\sin x\,(\cos x)^3} = \frac{1}{2(\cos x)^2} + \log(\tan x).$$

$$\int \frac{dx}{(\sin x)^2(\cos x)^4} = \frac{1}{3\sin x\,(\cos x)^3} - \frac{8}{3}\cot 2x.$$

$$\int \frac{dx}{(\sin x)^4(\cos x)^4} = -\frac{8\cos 2x}{3}\left\{\frac{1}{(\sin 2x)^3} + \frac{2}{\sin 2x}\right\}.$$

$$\int \frac{dx}{(\sin x)^5\cos x} = -\frac{1}{4(\sin x)^4} - \frac{1}{2(\sin x)^2} + \log(\tan x).$$

(17) If the function be $(\tan x)^n$ the formula of reduction is

$$\int dx\,(\tan x)^n = \frac{(\tan x)^{n-1}}{n-1} - \int dx\,(\tan x)^{n-2}.$$

comparing this with the given equation we find,

$$\phi'(y) = 0, \text{ and therefore } \phi(y) = C.$$

Hence the integral is

$$(x + y)^2 (y + z) = C.$$

(21) Let $dx(ay - bz) + dy(cz - ax) + dz(bx - cy) = 0.$

The integral is $\dfrac{ay - bz}{cz - ax} = C.$

(22) Let

$$(y^2 + yz + z^2)\,dx + (x^2 + xz + z^2)\,dy + (x^2 + xy + y^2)\,dz = 0.$$

The integral is $xy + xz + yz = a(x + y + z).$

(23) Let $dx + dy + dz + (x + y + z)\,dz = 0.$

The integral is $(x + y + z)\,\epsilon^z = C.$

(24) Let $z(x\,dx + y\,dy) + \{(x^2 + y^2)z - 1\}\,dz = 0.$

The integral is $\epsilon^z \dfrac{(x^2 + y^2)^{\frac{1}{2}}}{z} = C.$

SECT. 2. *Functions of an order higher than the first.*

Let $v = d^n u$ be a differential function involving x, y, and their differentials, and let x_1, x_2, x_3, &c. y_1, y_2, y_3, &c. be put for dx, d^2x, d^3x, &c. dy, d^2y, d^3y, &c. Then the conditions that v should be the differential of a function $d^{n-1}u$, are

$$\frac{dv}{dx} - d\,.\frac{dv}{dx_1} + d^2\,.\frac{dv}{dx_2} - d^3\,.\frac{dv}{dx_3} + \&\text{c.} = 0 \ldots (1),$$

$$\text{and} \quad \frac{dv}{dy} - d\,.\frac{dv}{dy_1} + d^2\,.\frac{dv}{dy_2} - d^3\,.\frac{dv}{dy_3} + \&\text{c.} = 0 \ldots (2).$$

The conditions that v should be the second differential of a function $d^{n-2}u$, are

$$\frac{dv}{dx_1} - 2d\,.\frac{dv}{dx_2} + 3d^2\,.\frac{dv}{dx_3} - \&\text{c.} = 0 \;\ldots\ldots\; (3),$$

$$\text{and} \quad \frac{dv}{dy_1} - 2d\,.\frac{dv}{dy_2} + 3d^2\frac{dv}{dy_3} - \&\text{c.} = 0 \;\ldots\ldots\; (4).$$

The conditions that v should be the third differential of a function $d^{n-3}u$ are,

$$\frac{dv}{dx_2} - 3d \, . \, \frac{dv}{dx_3} + 6d^2 \, . \, \frac{dv}{dx_4} - 10d^3 \frac{dv}{dx_5} + \&c. = 0 \ldots (5),$$

$$\frac{dv}{dy_2} - 3d \, . \, \frac{dv}{dy_3} + 6d^2 \frac{dv}{dy_4} - 10d^3 \frac{dv}{dy_5} + \&c. = 0 \ldots (6).$$

In a similar manner are found the conditions that v should be a differential of any order: the numerical coefficients follow the law of those of the Binomial Theorem in the case of a negative index.

These remarkable formulæ were first discovered by Euler (*Comm. Petrop.* Vol. VIII.) in his investigations concerning maxima and minima. A more direct demonstration is given by Condorcet, in his *Calcul. Integral.*

Ex. (1) Let $v = d^2u = xd^2y - yd^2x$.

Then $\dfrac{dv}{dx} = d^2y = y_2$, $\dfrac{dv}{dx_1} = 0$, $\dfrac{dv}{dx_2} = -y$.

Therefore the first equation of condition becomes

$$y_2 - d^2y = 0,$$

and is therefore satisfied. In the same way the second condition is also satisfied, and we find

$$du = xdy - ydx + C.$$

(2) Let $v = d^2u$

$$= x^2d^2y + (a+2)\,xdy\,dx + (ay + 2x)\,dx^2 + (a\,xy + x^2)\,d^2x.$$

Both the conditions (1) and (2) are satisfied in this case, and we find

$$du = x^2\,dy + axy\,dx + x^2\,dx + C.$$

(3) Let

$$v = d^2u = (ax - 2y)\,d^2y - 2dy^2 + 2a\,dy\,dx + ay\,d^2x.$$

In this case the conditions (3) and (4) are both satisfied, so that v is the second differential of a function, which is found to be

$$u = axy - y^2 + C.$$

(4) To find the condition that

$$R\,dx^2 + S\,dx\,dy + T\,dy^2$$

should admit of a first integral. If we assume $S = S_1 + S_2$ this may be put under the form

$$(R\,dx + S_1\,dy)\,dx + (S_2\,dx + T\,dy)\,dy;$$

and in order that it may admit of a first integral, we must have

$$\frac{d}{dy}(R\,dx + S_1\,dy) = \frac{d}{dx}(S_2\,dx + T\,dy),$$

or

$$\frac{dR}{dy}\,dx + \frac{dS_1}{dy}\,dy = \frac{dS_2}{dx}\,dx + \frac{dT}{dx}\,dy.$$

But from the indeterminateness of dx, and dy this involves the conditions

$$\frac{dR}{dy} = \frac{dS_2}{dx}, \qquad \frac{dT}{dx} = \frac{dS_1}{dy}.$$

Whence

$$\frac{d^2R}{dy^2} = \frac{d^2S_2}{dx\,dy}, \qquad \frac{d^2T}{dx^2} = \frac{d^2S_1}{dx\,dy};$$

and therefore

$$\frac{d^2R}{dy^2} + \frac{d^2T}{dx^2} = \frac{d^2S_1}{dx\,dy} + \frac{d^2S_2}{dx\,dy} = \frac{d^2S}{dx\,dy},$$

which is the required condition.

The complication of the formulæ when the order of the differentials rises above the second renders their application almost impracticable, and as the subject is not one of any practical importance, it is unnecessary to adduce other examples.

CHAPTER IV.

INTEGRATION OF DIFFERENTIAL EQUATIONS.

SECT. 1. *Linear Equations with constant coefficients.*

THESE form the largest class of Differential Equations which are integrable by one method, and they are of great importance, as many of the equations which are met with in the application of the Calculus to physics are either in this shape or may be reduced to it.

Let $$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \&c. + A_{n-1} \frac{dy}{dx} + A_n y = X, \quad (1)$$

be the general form of a linear differential equation with constant coefficients; A_1, $A_2 \ldots A_n$ being constants, and X being any function of x. On separating the symbols of operation from those of quantity this becomes

$$\left\{\left(\frac{d}{dx}\right)^n + A_1 \left(\frac{d}{dx}\right)^{n-1} + \&c\ldots + A_n\right\} y = X \text{ or } f\left(\frac{d}{dx}\right).y = X, \quad (2)$$

as we may write it for shortness. Now by the theorem given in Ex. 5. of Chap. xv. of the Differential Calculus, the complex operation $f\left(\frac{d}{dx}\right)$ is equivalent to

$$\left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\ldots\left(\frac{d}{dx} - a_n\right),$$

a_1, $a_2 \ldots a_n$ being the roots of the equation $f(x) = 0 \ldots\ldots (3)$

Hence performing on both sides of (2) the inverse process of

$$f\left(\frac{d}{dx}\right) \text{ or } \left(\frac{d}{dx} - a_1\right)\left(\frac{d}{dx} - a_2\right)\ldots\left(\frac{d}{dx} - a_n\right), \text{ we have}$$

$$y=\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}X=\left\{\left(\frac{d}{dx}-a_1\right)\left(\frac{d}{dx}-a_2\right)\dots\left(\frac{d}{dx}-a_n\right)\right\}^{-1}X. \quad (4).$$

The result of this transformation is different according to the nature of the roots of (3).

1st. Let all the roots be unequal; then by the theorem given in Ex. 6. Chap. xv. of the Differential Calculus, the equation (4) becomes

$$y=N_1\left(\frac{d}{dx}-a_1\right)^{-1}X+N_2\left(\frac{d}{dx}-a_2\right)^{-1}X+\dots$$
$$+N_n\left(\frac{d}{dx}-a_n\right)^{-1}X \dots\dots\dots (5)$$

where $N_1=\dfrac{1}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)}$,

and similarly for the other coefficients.

But by the theorem in Ex. 11. of the same chapter,

$$\left(\frac{d}{dx}-a_1\right)^{-1}X=\epsilon^{a_1x}\left(\frac{d}{dx}\right)^{-1}\epsilon^{-a_1x}X=\epsilon^{a_1x}\int dx\,\epsilon^{-a_1x}X.$$

A similar transformation being made of the other terms, we find

$$y=N_1\epsilon^{a_1x}\int dx\,\epsilon^{-a_1x}X+N_2\epsilon^{a_2x}\int dx\,\epsilon^{-a_2x}X+ \&c.$$
$$+N_n\epsilon^{a_nx}\int dx\,\epsilon^{-a_nx}X \dots\dots (6).$$

It is to be observed that each of the signs of integration would give rise to an arbitrary constant; and that this must be added in each of the terms when the integrations are effected. The value of y would then appear under the form

$$y=N_1\epsilon^{a_1x}\left(\int dx\,\epsilon^{-a_1x}X+C_1\right)+N_2\epsilon^{a_2x}\left(\int dx\,\epsilon^{-a_2x}X+C_2\right)+\&c.$$
$$+N_n\epsilon^{a_nx}\left(\int dx\,\epsilon^{-a_nx}X+C_n\right)\dots\dots(7).$$

C_1, $C_2 \dots C_n$ being the arbitrary constants.

The functions $C\epsilon^{ax}$ which arise in the integration are called *complementary functions.*

2nd. Let r of the roots of the equation (4) be equal to a. Then by the Theory of the decomposition of partial fractions we know that the factor $\left(\frac{d}{dx} - a\right)^r$ will give rise to a series of r terms in (5) of the form

$$M_p\left(\frac{d}{dx} - a\right)^{-p};$$

the coefficient M_p being equal to

$$\frac{1}{1 . 2 \dots (p-1)}\left(\frac{d}{dz}\right)^{p-1}\frac{(z-a)^r}{f(z)} \text{ when } z = a.$$

Now $\left(\frac{d}{dx} - a\right)^{-p} X = \epsilon^{ax} \int^p dx^p (\epsilon^{-ax} X)$

or, introducing the arbitrary constants which arise from the integration,

$$\left(\frac{d}{dx} - a\right)^{-p} X = \epsilon^{ax} \int^p dx^p (\epsilon^{-ax} X)$$
$$+ \epsilon^{ax}(C'_0 + C'_1 x + \text{\&c.} + C'_{p-1} x^{p-1}).$$

Therefore the complete value of y is

$$y = \epsilon^{ax}\{M_r \int^r dx^r (\epsilon^{-ax} X) + M_{r-1} \int^{r-1} dx^{r-1} (\epsilon^{-ax} X) + \text{\&c.}$$
$$+ M_1 \int dx (\epsilon^{-ax} X)\}$$
$$+ N_1 \epsilon^{a_1 x} \int dx (\epsilon^{-a_1 x} X) + N_2 \epsilon^{a_2 x} \int dx (\epsilon^{-a_2 x} X) + \text{\&c.}$$
$$+ N_{n-r} \epsilon^{a_{n-r} x} \int dx (\epsilon^{-a_{n-r} x} X)$$
$$+ \epsilon^{ax}(C'_0 + C'_1 x + C'_2 x^2 + \text{\&c.} + C'_{r-1} x^{r-1})$$
$$+ C_1 \epsilon^{a_1 x} + C_2 \epsilon^{a_2 x} + \text{\&c.} + C_{n-r} \epsilon^{a_{n-r} x}. \tag{8}$$

There are in all exactly n arbitrary constants as there ought to be.

3rd. Let there be a pair of impossible roots, which must be of the form

$$\alpha + (-)^{\frac{1}{2}}\beta \text{ and } \alpha - (-)^{\frac{1}{2}}\beta;$$

then the coefficients of the corresponding terms in (6) are of the forms

$$\frac{1}{A + (-)^{\frac{1}{2}} B} \text{ and } \frac{1}{A - (-)^{\frac{1}{2}} B}.$$

And as $\epsilon^{\{\alpha+(-)^{\frac{1}{2}}\beta\}x} = \epsilon^{\alpha x}\left\{\cos\beta x + (-)^{\frac{1}{2}}\sin\beta x\right\}$,
and $\epsilon^{\{\alpha-(-)^{\frac{1}{2}}\beta\}x} = \epsilon^{\alpha x}\left\{\cos\beta x - (-)^{\frac{1}{2}}\sin\beta x\right\}$.

The sum of the two corresponding terms in (6) is

$$\left.\begin{array}{c}\dfrac{2\epsilon^{\alpha x}(A\cos\beta x + B\sin\beta x)\int dx\,(\epsilon^{-\alpha x}\cos\beta x\,.\,X)}{A^2+B^2} \\ + \dfrac{2\epsilon^{\alpha x}(A\sin\beta x - B\cos\beta x)\int dx\,(\epsilon^{-\alpha x}\sin\beta x\,.\,X)}{A^2+B^2}\end{array}\right\} (9).$$

This may be put under a simpler form, for if

$$\frac{A}{(A^2+B^2)^{\frac{1}{2}}} = \cos\theta, \quad \frac{B}{(A^2+B^2)^{\frac{1}{2}}} = \sin\theta,$$

the sum of the terms becomes

$$\frac{2\epsilon^{\alpha x}\{\cos(\beta x-\theta)\int dx\,(\epsilon^{-\alpha x}\cos\beta x X) + \sin(\beta x-\theta)\int dx\,(\epsilon^{-\alpha x}\sin\beta x X)\}}{(A^2+B^2)^{\frac{1}{2}}} \quad (10).$$

The sum of the complementary functions

$$\epsilon^{\alpha x}\left\{C_1\epsilon^{(-)^{\frac{1}{2}}\beta x} + C_2\epsilon^{-(-)^{\frac{1}{2}}\beta x}\right\},$$

may evidently be put under the form

$$\epsilon^{\alpha x}(C\cos\beta x + C'\sin\beta x) = C\epsilon^{\alpha x}\cos(\beta x + \alpha). \quad (11)$$

If there be a number of equal pairs of impossible roots in the equation (3), the general expression for the value of y becomes so complicated as to be of little use, and it is therefore unnecessary to insert it here.

The preceding process may frequently be simplified in its application to particular cases, by means of the following considerations. The inverse operations are always reduced to the sum of several of the form $\left(\dfrac{d}{dx}+a\right)^{-r}X$ where r may be 1 or any positive integer. Now this operation will have a different effect according as it is expanded in ascending or descending powers of $\dfrac{d}{dx}$, that is, according as it is considered to be

$$\left(\frac{d}{dx}+a\right)^{-r}X \text{ or } \left(a+\frac{d}{dx}\right)^{-r}X,$$

inasmuch as in the one case it will involve integrals, while in the other it will involve differentials only. But a simple relation connects the two, for

$$\left(\frac{d}{dx}+a\right)^{-r}X=\left(a+\frac{d}{dx}\right)^{-r}X+\left(\frac{d}{dx}+a\right)^{-r}0.$$

Now the latter term $\left(\frac{d}{dx}+a\right)^{-r}0=\epsilon^{-ax}\int^r dx^r\,0$ $=\epsilon^{-ax}(C_0+C_1x+C_2x^2+\&c.+C_{r-1}x^{r-1})$, and the former term $\left(a+\frac{d}{dx}\right)^{-r}X$, being expanded in ascending powers of $\frac{d}{dx}$, will give rise to a series of differentials which are always easily found, and which, when X is a rational and integral function of x of n dimensions, always breaks off at the $(n+1)^{\text{th}}$ term. But since each factor of the form $\left(\frac{d}{dx}+a\right)^{-r}$ gives rise to a separate complementary function, while X is operated on by all in succession, it is sufficient to expand $\left\{f\left(\frac{d}{dx}\right)\right\}^{-1}$ in descending powers of X, without splitting it into its binomial factors, and then to add the complementary functions corresponding to each of these factors.

If the function X be of the form ϵ^{mx}, the result of the operation $f\left(\frac{d}{dx}\right)\epsilon^{mx}$ takes a very simple shape. For if we expand $f\left(\frac{d}{dx}\right)$ in ascending powers of $\frac{d}{dx}$ so as to have a series of the form

$$\left\{A+B\frac{d}{dx}+C\left(\frac{d}{dx}\right)^2+D\left(\frac{d}{dx}\right)^3+\&c.\right\}\epsilon^{mx},$$

and then operate on ϵ^{mx} with each term separately, we find, as $\left(\frac{d}{dx}\right)^r\epsilon^{mx}=m^r\epsilon^{mx}$, that the series becomes

$$\{A+Bm+Cm^2+Dm^3+\&c.\}\,\epsilon^{mx}=f(m)\,\epsilon^{mx}.$$

Hence for example, we have $\left(\frac{d}{dx}-a\right)^r\epsilon^{mx}=\frac{\epsilon^{mx}}{(m-a)^r}$.

If the function X be of the form $\cos mx$ or $\sin mx$, and if the operating function be $f\left(\frac{d^2}{dx^2}\right)$, it is easy to see by the same method that, as $\left(\frac{d}{dx}\right)^2 \cos mx = -m^2 \cos mx$, and $\left(\frac{d}{dx}\right)^2 \sin mx = -m^2 \sin mx$,

$$f\left(\frac{d^2}{dx^2}\right) \cos mx = f(-m^2) \cos mx$$

$$\text{and } f\left(\frac{d^2}{dx^2}\right) \sin mx = f(-m^2) \sin mx.$$

The preceding theory may be stated in the form of the following proposition: if the integral of an equation

$$\left\{f\left(\frac{d}{dx}\right)\right\} y = 0$$

be given, that of the equation

$$\left\{f\left(\frac{d}{dx}\right)\right\} y = X$$

can be found from it by differentiation only.

Ex. (1) Let $\frac{dy}{dx} - ay = x^4$;

therefore $\left(\frac{d}{dx} - a\right) y = x^4$,

$$\text{and } y = \left(\frac{d}{dx} - a\right)^{-1} x^4$$

$$= -\left(a - \frac{d}{dx}\right)^{-1} x^4 + C\epsilon^{ax}$$

$$= -\frac{1}{a}\left\{1 + \frac{1}{a}\left(\frac{d}{dx}\right) + \frac{1}{a^2}\left(\frac{d}{dx}\right)^2 + \frac{1}{a^3}\left(\frac{d}{dx}\right)^3 + \frac{1}{a^4}\left(\frac{d}{dx}\right)^4\right\} x^4 + C\epsilon^{ax},$$

the differentials after the fourth being neglected. Effecting the differentiations

$$y = C\epsilon^{ax} - \frac{x^4}{a} - \frac{4x^3}{a^2} - \frac{4.3.x^2}{a^3} - \frac{4.3.2.x}{a^4} - \frac{4.3.2.1}{a^5}.$$

(2) Let $\frac{dy}{dx} + ay = \epsilon^{mx}$;

therefore $y = \left(\frac{d}{dx} + a\right)^{-1} \epsilon^{mx} = \frac{\epsilon^{mx}}{a + m} + C\epsilon^{-ax}.$

(3) Let $\frac{dy}{dx} - ay = \epsilon^{mx} \cos rx,$

$$y = \left(\frac{d}{dx} - a\right)^{-1} (\epsilon^{mx} \cos rx) = \epsilon^{ax} \int dx \left\{\epsilon^{(m-a)x} \cos rx\right\};$$

therefore $y = \epsilon^{mx} \frac{\{(m - a) \cos rx + r \sin rx\}}{(m - a)^2 + r^2} + C\epsilon^{ax}.$

(4) Let $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \frac{x}{(1 + x)^2}.$

Hence $\left(\frac{d}{dx} + 2\right)\left(\frac{d}{dx} + 1\right) y = \frac{x}{(1 + x)^2};$

therefore $y = -\epsilon^{-2x} \int dx \frac{\epsilon^{2x} x}{(1 + x)^2} + \epsilon^{-x} \int dx \frac{\epsilon^x x}{(1 + x)^2}.$

But $\int dx \frac{\epsilon^x x}{(1 + x)^2} = \int dx\, \epsilon^x \left(\frac{1}{1 + x} - \frac{1}{(1 + x)^2}\right) = \frac{\epsilon^x}{1 + x};$

and integrating by parts, we find

$$\int dx \frac{\epsilon^{2x} x}{(1 + x)^2} = \frac{\epsilon^{2x}}{1 + x} - \int dx \frac{\epsilon^{2x}}{1 + x};$$

therefore $y = \epsilon^{-2x} \int dx \frac{\epsilon^{2x}}{1 + x} + C_1 \epsilon^{-x} + C_2 \epsilon^{-2x}.$

(5) Let $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2.$

This is equivalent to

$$\left(\frac{d}{dx} - 2\right)^2 y = x^2;$$

therefore $y = \left(\frac{d}{dx} - 2\right)^{-2} x^2$

$$= \left(2 - \frac{d}{dx}\right)^{-2} x^2 + \epsilon^{2x}(C + C_1 x)$$

$$= \frac{1}{2^2}\left\{1 + \frac{2}{2^2}\frac{d}{dx} + \frac{3}{2^2}\left(\frac{d}{dx}\right)^2\right\} x^2 + \epsilon^{2x}(C + C_1 x).$$

Hence, performing the differentiations,

$$y = \frac{x^2}{2^2} + \frac{4x}{2^3} + \frac{6}{2^4} + \epsilon^{2x}(C + C_1 x).$$

(6) Let $\dfrac{d^2 y}{dx^2} - 2m\dfrac{dy}{dx} + m^2 y = \sin nx.$

This gives $\left(\dfrac{d}{dx} - m\right)^2 y = \sin nx;$

and therefore $y = \left(\dfrac{d}{dx} - m\right)^{-2} \sin nx$

$$= \epsilon^{mx} \textstyle\int^2 dx^2 (\epsilon^{-mx} \sin nx) + \epsilon^{mx}(C + C_1 x).$$

Let $m = (m^2 + n^2)^{\frac{1}{2}} \cos\theta$, $n = (m^2 + n^2)^{\frac{1}{2}} \sin\theta$; then

$$\textstyle\int^2 dx^2 (\epsilon^{-mx} \sin nx) = \epsilon^{-mx} \dfrac{\sin(nx + 2\theta)}{m^2 + n^2};$$

and therefore $y = \dfrac{\sin(nx + 2\theta)}{m^2 + n^2} + \epsilon^{mx}(C + C_1 x).$

(7) Let $\dfrac{d^2 y}{dx^2} + n^2 y = 0.$

This contains two impossible factors

$$\left[\left\{\frac{d}{dx} + (-)^{\frac{1}{2}} n\right\}\left\{\frac{d}{dx} - (-)^{\frac{1}{2}} n\right\}\right] y = 0;$$

therefore by the formula (11)

$$y = C\cos nx + C' \sin nx = C_1 \cos(nx + a).$$

The same result may be obtained by a different process, which is subjoined as it points out very distinctly the reason why these circular functions appear in the integral.

$$y = \left\{\left(\frac{d}{dx}\right)^2 + n^2\right\}^{-1} 0 = \left(\frac{d}{dx}\right)^{-2} \left\{1 + n^2 \left(\frac{d}{dx}\right)^{-2}\right\}^{-1} 0.$$

It is indifferent in which order we perform the operations; taking then $\left(\dfrac{d}{dx}\right)^{-2}$ first, we have, as $\left(\dfrac{d}{dx}\right)^{-2} 0 = C + C_1 x$,

$$y = \left\{1 + n^2 \left(\frac{d}{dx}\right)^{-2}\right\}^{-1} (C + C_1 x).$$

Expanding the operative symbol,

$$y = \left\{1 - n^2 \left(\frac{d}{dx}\right)^{-2} + n^4 \left(\frac{d}{dx}\right)^{-4} - n^6 \left(\frac{d}{dx}\right)^{-6} + \&c.\right\}(C + C_1 x)$$

$$= C\left\{1 - \frac{n^2 x^2}{1.2} + \frac{n^4 x^4}{1.2.3.4} - \frac{n^6 x^6}{1.2.3.4.5.6} + \&c.\right\}$$

$$+ C_1\left\{x - \frac{n^2 x^3}{1.2.3} + \frac{n^4 x^5}{1.2.3.4.5} - \frac{n^6 x^7}{1.2.3.4.5.6.7} + \&c.\right\}$$

$$= C \cos nx + \frac{C_1}{n} \sin nx = C \cos nx + C' \sin nx,$$

as the constant is arbitrary.

As operating factors of the form $\left(\frac{d}{dx}\right)^2 + n^2$ very frequently occur in differential equations, it is convenient to keep in mind that the complementary function due to it is of the form $C \cos nx + C' \sin nx$.

(8) Let $\frac{d^2 y}{dx^2} + n^2 y = \cos mx$.

Then $y = \left\{\left(\frac{d}{dx}\right)^2 + n^2\right\}^{-1} \cos mx + C \cos nx + C' \sin nx$

$$= \frac{\cos mx}{n^2 - m^2} + C \cos nx + C' \sin nx.$$

(9) Let $\frac{d^4 y}{dx^4} + 5\frac{d^2 y}{dx^2} + 6y = \sin mx$.

That is $\left\{\left(\frac{d}{dx}\right)^2 + 2\right\}\left\{\left(\frac{d}{dx}\right)^2 + 3\right\} y = \sin mx$;

therefore

$$y = \frac{\sin mx}{m^4 - 5m^2 + 6} + C \cos (2^{\frac{1}{2}} x + \alpha) + C_1 \cos (3^{\frac{1}{2}} x + \beta).$$

(10) Let $\frac{d^2 y}{dx^2} + y = x^n$.

Then $y = \left\{1 + \left(\frac{d}{dx}\right)^2\right\}^{-1} x^n + C \cos x + C_1 \sin x$

$$y = x^n - n(n-1)x^{n-2} + n(n-1)(n-2)(n-3)x^{n-4} - \&c.$$
$$+ C \cos x + C_1 \sin x.$$

(11) Let $\dfrac{d^4y}{dx^4} - a^4y = x^3$.

The roots of $z^4 - a^4 = 0$, are

$$z = +a, \quad z = -a, \quad z = +(-)^{\frac{1}{2}}a, \quad z = -(-)^{\frac{1}{2}}a;$$

therefore $y = -\dfrac{x^3}{a^4} + C\epsilon^{ax} + C_1\epsilon^{-ax} + C_2\cos(ax + \beta)$.

(12) Let $\dfrac{d^4y}{dx^4} + 2a^2\dfrac{d^2y}{dx^2} + a^4y = \cos x$.

This is equivalent to

$$\left\{\left(\frac{d}{dx}\right)^2 + a^2\right\}^2 y = \cos x.$$

From which we find

$$y = \frac{\cos x}{(a^2-1)^2} + (C + C_1x)\cos ax + (C' + C_1'x)\sin ax.$$

(13) Let

$$\frac{d^5y}{dx^5} - 2\frac{d^4y}{dx^4} + 5\frac{d^3y}{dx^3} - 10\frac{d^2y}{dx^2} - 36\frac{dy}{dx} + 72y = \epsilon^{mx}.$$

When the differential expression is divided into factors, this may be put under the form

$$\left(\frac{d}{dx} - 2\right)^2\left(\frac{d}{dx} + 2\right)\left\{\left(\frac{d}{dx}\right)^2 + 9\right\}y = \epsilon^{mx}.$$

Whence we find

$$y = \frac{\epsilon^{mx}}{(m-2)^2(m+2)(m^2+9)} + \epsilon^{2x}(C + C_1x) + C_2\epsilon^{-2x} + C_3\cos(3x + \alpha).$$

(14) Let $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 15y = x^2$.

The integral of this is

$$y = \frac{x^2}{15} + \frac{2x}{(15)^2} - \frac{28}{(15)^3} + \epsilon^{x}(C\sin 2x + C_1\cos 2x) + C_3\epsilon^{-3x}.$$

(15) Let $\dfrac{d^{2n}y}{dx^{2n}} - a^{2n}y = X$.

The roots of the equation

$$x^{2n} - a^{2n} = 0,$$

are included in the formula

$$a\left\{\cos\phi \pm (-)^{\frac{1}{2}}\sin\phi\right\},$$

where $\phi = \dfrac{\lambda\pi}{n}$, λ receiving all values from 0 to n; and the roots corresponding to $\lambda = 0$ and $\lambda = n$ being $+a$ and $-a$.

If now we take a pair of the impossible roots, which we may call α and β, the corresponding terms in the general value of y are

$$\left\{N_1\left(\frac{d}{dx} - \alpha\right)^{-1} + N_2\left(\frac{d}{dx} - \beta\right)^{-1}\right\}X.$$

But by the theory of the decomposition of rational fractions we know that

$$N_1 = \frac{1}{2n\alpha^{2n-1}} = \frac{\alpha}{2na^{2n}} = \frac{\cos\phi + (-)^{\frac{1}{2}}\sin\phi}{2na^{2n-1}}.$$

Similarly $N_2 = \dfrac{\cos\phi - (-)^{\frac{1}{2}}\sin\phi}{2na^{2n-1}}$.

Now

$$\left(\frac{d}{dx} - \alpha\right)^{-1}X = \epsilon^{ax\cos\phi}\left\{\cos(ax\sin\phi) + (-)^{\frac{1}{2}}\sin(ax\sin\phi)\right\}$$
$$\times\int dx\, X\epsilon^{-ax\cos\phi}\left\{\cos(ax\sin\phi) - (-)^{\frac{1}{2}}\sin(ax\sin\phi)\right\},$$

and

$$\left(\frac{d}{dx} - \beta\right)^{-1}X = \epsilon^{ax\cos\phi}\left\{\cos(ax\sin\phi) - (-)^{\frac{1}{2}}\sin(ax\sin\phi)\right\}$$
$$\times\int dx\, X\epsilon^{-ax\cos\phi}\left\{\cos(ax\sin\phi) + (-)^{\frac{1}{2}}\sin(ax\sin\phi)\right\}.$$

Therefore substituting these expressions, the two terms in the value of y become after reduction,

$$\frac{1}{na^{2n-1}}\,\epsilon^{ax\cos\phi}\cos(ax\sin\phi + \phi)\int dx\left\{X\epsilon^{-ax\cos\phi}\cos(ax\sin\phi)\right\}$$

$$+\frac{1}{na^{2n-1}}\,\epsilon^{ax\cos\phi}\sin(ax\sin\phi + \phi)\int dx\left\{X\epsilon^{-ax\cos\phi}\sin(ax\sin\phi)\right\}.$$

Also the two roots $+a$ and $-a$ give rise to the terms

$$\frac{1}{2na^{2n-1}}\left(\epsilon^{-ax}\int dx\,X\epsilon^{ax} - \epsilon^{ax}\int dx\,X\epsilon^{-ax}\right).$$

Hence we put the expression for y under the form

$$\begin{aligned}
y = {} & \frac{1}{2na^{2n-1}}\left(\epsilon^{-ax}\int dx X\epsilon^{ax} - \epsilon^{ax}\int dx X\epsilon^{-ax}\right) \\
& + \frac{1}{na^{2n-1}}\,\Sigma\left[\epsilon^{ax\cos\phi}\cos(ax\sin\phi+\phi)\times\right. \\
& \qquad\qquad \left.\int dx\left\{X\epsilon^{-ax\cos\phi}\cos(ax\sin\phi)\right\}\right] \\
& + \frac{1}{na^{2n-1}}\,\Sigma\left[\epsilon^{ax\cos\phi}\sin(ax\sin\phi+\phi)\times\right. \\
& \qquad\qquad \left.\int dx\left\{X\epsilon^{-ax\cos\phi}\sin(ax\sin\phi)\right\}\right].
\end{aligned}$$

The symbol Σ implies the sum of terms derived from assigning to ϕ in the preceding expression all values from $\frac{\pi}{n}$ to $(n-1)\frac{\pi}{n}$.

The complementary functions are, for the sake of shortness, supposed to be included in the signs of integration; but if we wish to see their form, we have only to make $X=0$ in the preceding expression when it becomes

$$\begin{aligned}
y = {} & C_1\epsilon^{-ax} + C_2\epsilon^{ax} \\
& + \epsilon^{a\cos\frac{\pi}{n}x}\left\{C_3\cos\left(ax\sin\frac{\pi}{n}+\frac{\pi}{n}\right) + C_4\sin\left(ax\sin\frac{\pi}{n}+\frac{\pi}{n}\right)\right\} \\
& + \epsilon^{a\cos\frac{2\pi}{n}x}\left\{C_5\cos\left(ax\sin\frac{2\pi}{n}+\frac{2\pi}{n}\right) + C_6\sin\left(ax\sin\frac{2\pi}{n}+\frac{2\pi}{n}\right)\right\} \\
& + \&c. \qquad\qquad\qquad + \&c.
\end{aligned}$$

This is evidently the solution of the equation

$$\frac{d^{2n}y}{dx^{2n}} - a^{2n}y = 0.$$

Euler, *Calc. Integ.* Vol. II. Sect. 2, Cap. IV.

(16) Let $\frac{d^6y}{dx^6} - a^6y = \cos mx$.

$$\text{then } y = -\frac{\cos mx}{a^6 + m^6} + C_1\epsilon^{-ax} + C_2\epsilon^{ax}$$

$$+ \epsilon^{\frac{1}{2}ax}\left\{C_3\cos\left(ax\frac{3^{\frac{1}{2}}}{2} + \frac{\pi}{3}\right) + C_4\sin\left(ax\frac{3^{\frac{1}{2}}}{2} + \frac{\pi}{3}\right)\right\}$$

$$+ \epsilon^{-\frac{1}{2}ax}\left\{C_5\cos\left(ax\frac{3^{\frac{1}{2}}}{2} + \frac{2\pi}{3}\right) + C_6\sin\left(ax\frac{3^{\frac{1}{2}}}{2} + \frac{2\pi}{3}\right)\right\}.$$

(17) Let $a^n y + n a^{n-1}\dfrac{dy}{dx} + \dfrac{n(n-1)}{1.2}a^{n-2}\dfrac{d^2y}{dx^2} + \&c. = X$; n being a positive integer.

Here the operating function is $\left(a + \dfrac{d}{dx}\right)^n$, which is composed of n equal factors; consequently

$$y = \left(a + \frac{d}{dx}\right)^{-n} X$$

$$= \epsilon^{-ax}\int^n dx^n\,\epsilon^{ax}X + \epsilon^{-ax}(C_0 + C_1x + C_2x^2 + \&c. + C_{n-1}x^{n-1}).$$

The term $\epsilon^{-ax}\int^n dx^n\,\epsilon^{ax}X$ may either be integrated by successive steps, or by the general formula for integration by parts; or what will generally be more convenient, the function $\left(a + \dfrac{d}{dx}\right)^{-n}$ may be expanded in ascending powers of $\dfrac{d}{dx}$.

If n were negative or fractional, the first term would retain the same form, but the form of the complementary function would be different from the difference between the roots of

$$(x + a)^n = 0,$$

when n is integer and when it is fractional or negative. I cannot however here enter into a discussion of the difficulties of this subject, which is closely connected with that of General Differentiation. Euler, *Calc. Integ.* Ib.

(18) Let $\dfrac{d^n y}{dx^n} + \dfrac{d^{n-1}y}{dx^{n-1}} + \&c. + \dfrac{dy}{dx} + y = X.$

This may be put under the form

$$\frac{\left(\frac{d}{dx}\right)^{n+1}-1}{\frac{d}{dx}-1}y=X.$$

Now the factors of $\dfrac{\left(\frac{d}{dx}\right)^{n+1}-1}{\frac{d}{dx}-1}$ are the same as those of $\left(\dfrac{d}{dx}\right)^{n+1}-1$, omitting $\dfrac{d}{dx}-1$; therefore if $n+1$ be odd, the integral of the preceding equation consists of a number of terms of the form

$$\frac{4}{n+1}\sin\tfrac{1}{2}\theta\,\epsilon^{x\cos\theta}\cos\tfrac{1}{2}(3\theta+2x\sin\theta)\int dx\,\epsilon^{-x\cos\theta}X\sin(x\sin\theta)$$

$$-\frac{4}{n+1}\sin\tfrac{1}{2}\theta\,\epsilon^{x\cos\theta}\sin\tfrac{1}{2}(3\theta+2x\sin\theta)\int dx\,\epsilon^{-x\cos\theta}X\cos(x\sin\theta);$$

where θ receives all values $\dfrac{2\pi}{n+1}$, $\dfrac{4\pi}{n+1}$, $\dfrac{6\pi}{n+1}$, &c. which are less than π.

If $n+1$ be even we must add the term

$$\frac{2}{n+1}\epsilon^{-x}\int dx\,\epsilon^{x}X.$$

Euler, *Calc. Integ.* Ib.

(19) Let $y-\dfrac{1}{1.2}\dfrac{d^2y}{dx^2}+\dfrac{1}{1.2.3.4}\dfrac{d^4y}{dx^4}-\text{\&c.}=X,$

or $\cos\left(\dfrac{d}{dx}\right)y=X.$

The roots of the equation

$$\cos x=0,$$

are $\pm\dfrac{\pi}{2}$, $\pm\dfrac{3\pi}{2}$, $\pm\dfrac{5\pi}{2}$, &c.,

and the factors of $\cos\left(\frac{d}{dx}\right)$ are therefore

$$\left(1-\frac{2}{\pi}\frac{d}{dx}\right)\left(1+\frac{2}{\pi}\frac{d}{dx}\right)\left(1-\frac{2}{3\pi}\frac{d}{dx}\right)\left(1+\frac{2}{3\pi}\frac{d}{dx}\right), \text{ \&c.}$$

Hence decomposing $\left\{\cos\left(\frac{d}{dx}\right)\right\}^{-1}$ into partial fractions, we find

$$y=\left(\frac{\pi}{2}-\frac{d}{dx}\right)^{-1}X+\left(\frac{\pi}{2}+\frac{d}{dx}\right)^{-1}X-\left(\frac{3\pi}{2}-\frac{d}{dx}\right)^{-1}.X$$
$$-\left(\frac{3\pi}{2}+\frac{d}{dx}\right)^{-1}X+\text{\&c.}$$

and therefore

$$\begin{aligned}y = &-\epsilon^{\frac{\pi}{2}x}\int dx\,\epsilon^{-\frac{\pi}{2}x}X+\epsilon^{-\frac{\pi}{2}x}\int dx\,\epsilon^{\frac{\pi}{2}x}X\\
&+\epsilon^{\frac{3\pi}{2}x}\int dx\,\epsilon^{-\frac{3\pi}{2}x}X-\epsilon^{-\frac{3\pi}{2}x}\int dx\,\epsilon^{\frac{3\pi}{2}x}X\\
&-\epsilon^{\frac{5\pi}{2}x}\int dx\,\epsilon^{-\frac{5\pi}{2}x}X+\epsilon^{-\frac{5\pi}{2}x}\int dx\,\epsilon^{\frac{5\pi}{2}x}X\\
&+\qquad\text{\&c.}\qquad-\qquad\text{\&c.}\\
&+C_1\epsilon^{\frac{\pi}{2}x}+C_3\epsilon^{\frac{3\pi}{2}x}+C_5\epsilon^{\frac{5\pi}{2}x}+\text{\&c.}\\
&+C_1'\epsilon^{-\frac{\pi x}{2}}+C_3'\epsilon^{-\frac{3\pi}{2}x}+C_5'\epsilon^{-\frac{5\pi}{2}x}+\text{\&c.}\end{aligned}$$

Euler, *Calc. Integ.* Ib.

(20) Let the equation be

$$y+\frac{n(n-1)}{1.2}\frac{d^2y}{dx^2}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}\frac{d^4y}{dx^4}+\text{\&c.}=X.$$

The factors of the operating function in this case are the same as those of the algebraical function

$$\tfrac{1}{2}\left\{(1+z)^n+(1-z)^n\right\}.$$

The quadratic factors of this expression are given by the formula

$$(1+z)^2-2(1-z^2)\cos 2\theta+(1-z)^2,$$

where $\theta=\dfrac{(2r+1)\pi}{2n}$.

From this we easily find the simple factors of the operating function to be

$$\frac{d}{dx} \pm (-)^{\frac{1}{2}} \tan\theta;$$

Therefore decomposing it into partial fractions, as in the previous Examples, we find that y consists of a number of terms of the form

$$\pm \frac{2(\cos\theta)^{n-1}}{n} \left\{ \begin{array}{l} \sin(x\tan\theta) \int dx\, X \cos(x\tan\theta) \\ -\cos(x\tan\theta) \int dx\, X \sin(x\tan\theta) \end{array} \right\};$$

θ receiving the values $\frac{\pi}{2n}$, $\frac{3\pi}{2n}$, $\frac{5\pi}{2n}$, &c., so long as they are less than $\frac{\pi}{2}$. Euler, *Calc. Integ.* Ib.

It sometimes happens that the inverse processes, such as $\left(\frac{d}{dx} - a\right)^{-1} X$, fail, from the coefficients becoming infinite, in the same way as the formula for integrating x^n fails when $n = -1$. Thus for instance,

$$\left(\frac{d}{dx} - a\right)^{-1} \epsilon^{mx} = \frac{\epsilon^{mx}}{m-a} = \infty \text{ when } m = a.$$

The method to be adopted in such cases is the same in principle as that used for determining the value of $\int \frac{dx}{x}$. It is this: since the function becomes infinite in these cases, we so assume the arbitrary constant in the complementary function as to make the formula assume the indeterminate form $\frac{0}{0}$, the true value of which may be easily determined by the ordinary rules. The assumption made with respect to the arbitrary constant is that it shall be negative and infinite, so that the difference of the two infinite quantities may be finite.

(21) Let the equation be

$$\frac{dy}{dx} - ay = \epsilon^{ax}.$$

The solution of this by the usual formula would be

$$y = \frac{\epsilon^{ax}}{a-a} + C\epsilon^{ax}.$$

To determine the real value of this, let us take the equation

$$\frac{dy}{dx} - ay = \epsilon^{mx},$$

the integral of which is

$$y = \frac{\epsilon^{mx}}{m-a} + C\epsilon^{ax}.$$

Now C being an arbitrary constant, we may assume it to be equal to

$$C_1 - \frac{1}{m-a},$$

so that

$$y = \frac{\epsilon^{mx} - \epsilon^{ax}}{m-a} + C_1\epsilon^{ax}.$$

When $m = a$, the first term of this becomes $\frac{0}{0}$; and its true value is easily seen, by differentiating numerator and denominator with respect to m, to be $x\epsilon^{ax}$ when $m = a$. Therefore

$$y = x\epsilon^{ax} + C_1\epsilon^{ax}$$

is the solution of the equation

$$\frac{dy}{dx} - ay = \epsilon^{ax}.$$

(22) Let the equation be

$$\left(\frac{d}{dx} - a\right)^r y = \epsilon^{ax}.$$

The solution of this by the usual method would be

$$y = \frac{\epsilon^{ax}}{(a-a)^r} + \epsilon^{ax}(C_0 + C_1x + \&\text{c.} + C_{r-1}x^{r-1}),$$

which is a nugatory result. Proceeding by the same method as in the last example, we find the function whose true value is to be determined to be

$$\frac{\epsilon^{mx} - \epsilon^{ax}}{(m-a)^r} \text{ when } m = a.$$

By Ex. 15. of Chap. VI. of the *Diff. Calc.* we find the true value of this to be

$$\frac{x^r \epsilon^{ax}}{1.2.3 \ldots (r-1) r};$$

therefore

$$y = \frac{x^r \epsilon^{ax}}{1.2.3 \ldots (r-1) r} + \epsilon^{ax} (C_0 + C_1 x + \&c. + C_{r-1} x^{r-1}),$$

is the solution of the given equation.

(23) Let the equation be

$$\frac{d^2y}{dx^2} + n^2 y = \cos nx.$$

The solution of this by the usual rule would be

$$y = \frac{\cos nx}{n^2 - n^2} + C \cos nx + C_1 \sin nx.$$

If we assume $C = C' - \dfrac{1}{n^2 - m^2}$ we have to find the true value of the function

$$\frac{\cos mx - \cos nx}{n^2 - m^2}, \quad \text{when } m = n.$$

This is easily seen to be

$$\frac{x \sin nx}{2n},$$

so that the solution of the given equation is

$$y = \frac{x \sin nx}{2n} + C' \cos nx + C_1 \sin nx.$$

This example is one of great importance, for in the application of analysis to physics, equations of this form frequently occur; and as the value of y is not simply periodic, but admits of indefinite increase, it indicates a change in the physical circumstances of the problem. Cases of this kind occur in the theory of the disturbed motions of pendulums and of the Lunar perturbations.

(24) If the equation be

$$\left\{\left(\frac{d}{dx}\right)^2 + n^2\right\}^r y = \cos nx,$$

we shall find by means of Ex. 21, of Chap. VI. of the Diff. Calc. that the integral is

$$y = \frac{(-)^r x^r \cos\left(nx + r\dfrac{\pi}{2}\right)}{(2n)^r 1.2.3 \ldots r}.$$

SECT. 2. *Equations in which the coefficients are functions of the independent variable.*

Equations of this class cannot be generally integrated by one method, but a considerable number may be reduced to the class discussed in the preceding section.

I. In the first place, all equations of the first order may be reduced to equations with constant coefficients by a change of the independent variable, or by some equivalent process. The general form of a linear equation of the first order is

$$\frac{dy}{dx} + Py = X,$$

P and X being functions of x. Assume

$$dt = P dx, \text{ so that } t = \int P dx;$$

then the equation becomes

$$\frac{dy}{dt} + y = \frac{X}{P};$$

the integral of which is by the preceding section,

$$y = \epsilon^{-t} \int dt \frac{X}{P} \epsilon^t + C\epsilon^{-t},$$

or putting for t its value

$$y = \epsilon^{-\int P dx} \int dx\, X \epsilon^{\int P dx} + C\epsilon^{-\int P dx};$$

which is the complete solution of the equation.

(1) Let the equation be

$$(1 - x^2)\frac{dy}{dx} + xy = ax;$$

Here $Pdx = \dfrac{xdx}{1 - x^2}$ and $\int Pdx = -\log (1 - x^2)^{\frac{1}{2}}$.

Therefore $y = a + C(1 - x^2)^{\frac{1}{2}}$.

(2) Let $(1 + x^2)^{\frac{1}{2}} \dfrac{dy}{dx} + ny = a(1 + x^2)^{\frac{1}{2}}$.

Here $Pdx = \dfrac{ndx}{(1 + x^2)^{\frac{1}{2}}}$, $\int Pdx = n \log \{x + (1 + x^2)^{\frac{1}{2}}\}$,

and the solution is

$$y = \frac{a}{2(n + 1)} \{(1 + x^2)^{\frac{1}{2}} + x\} + \frac{a}{2(n - 1)} \{(1 + x^2)^{\frac{1}{2}} - x\}$$
$$+ C\{(1 + x^2)^{\frac{1}{2}} - x\}^n.$$

(3) Let $\dfrac{dy}{dx} + \dfrac{ay}{1 - x^2} = \dfrac{1 + x}{(1 - x)^3}$.

The integral of this is

$$y = \frac{1}{a + 4} \left(\frac{1 + x}{1 - x}\right)^2 + C\left(\frac{1 - x}{1 + x}\right)^{\frac{a}{2}}.$$

(4) Let $(1 - x^2)^{\frac{1}{2}} \dfrac{dy}{dx} - ny = x(1 - x^2)^{\frac{1}{2}}$.

The integral of this is

$$y = -\frac{\{nx(1 - x^2)^{\frac{1}{2}} + 1 - 2x^2\}}{n^2 + 4} + Ce^{n \sin^{-1} x}.$$

II. Equations of all orders of the form

$$(a + bx)^n \frac{d^n y}{dx^n} + A_1 (a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \text{\&c.} + A_n y = X,$$

where $A_1, A_2, \ldots\ldots A_n$ are constants, can always be integrated by a change of the independent variable.

In the first place, if we assume

$$a + bx = bz,$$

the equation evidently takes the form

$$b^n z^n \frac{d^n y}{dz^n} + A_1 b^{n-1} z^{n-1} \frac{d^{n-1} y}{dz^{n-1}} + \text{\&c.} + A_n y = Z;$$

where Z is what X becomes, when we substitute in it $z-\frac{a}{b}$ for x. As b^n, b^{n-1}, &c. are constants, this equation may, by dividing by b^n, be put under the form

$$z^n\frac{d^ny}{dz^n}+A_1'z^{n-1}\frac{d^{n-1}y}{dz^{n-1}}+\text{&c.}+A_n'y=Z',$$

where $A_1'=\frac{A_1}{b}$, $A_2'=\frac{A_2}{b^2}$, &c. and $Z'=\frac{Z}{b^n}$.

In this equation make $\frac{dz}{z}=dt$, or $z=\epsilon^t$. Then by Ex. 6. of Chap. III. of the Diff. Calc. we have

$$z^r\frac{d^ry}{dz^r}=\frac{d}{dt}\left(\frac{d}{dt}-1\right)\left(\frac{d}{dt}-2\right)\dots\left(\frac{d}{dt}-r+1\right),$$

so that the substitution of t for z will give rise to an equation of the form

$$\frac{d^ny}{dt^n}+B_1\frac{d^{n-1}y}{dt^{n-1}}+B_2\frac{d^{n-2}y}{dt^{n-2}}+\text{&c.}+B_ny=T.$$

where T is what Z' becomes when we substitute in it ϵ^t for z. The coefficients B_1, B_2, &c. are constant, so that this equation is integrable by the method given in the last section. This transformation was first given by Legendre, *Mémoires de l'Académie*, 1787, p. 336.

(5) Let $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^m.$$

Making $x=\epsilon^t$, the transformed equation is

$$\frac{d^2y}{dt^2}-y=\epsilon^{mt};$$

the integral of which is

$$y=\frac{\epsilon^{mt}}{m^2-1}+C\epsilon^t+C_1\epsilon^{-t},$$

or $$y=\frac{x^m}{m^2-1}+Cx+\frac{C_1}{x}.$$

(6) Let $$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$$

Changing the independent variable from x to t, and making $x = \epsilon^t$, this becomes

$$\left(\frac{d}{dt} + 1\right)^2 y = \frac{1}{(1-\epsilon^t)^2};$$

the integral of which is

$$y = \epsilon^{-t}\int^2 dt^2 \frac{\epsilon^t}{(1-\epsilon^t)^2} + (C + C_1 t)\,\epsilon^{-t}.$$

Therefore $$y = \log\left(\frac{x}{1-x}\right)^{\frac{1}{x}} + \frac{1}{x}(C + C_1 \log x).$$

(7) Let

$$(1+x)^3\frac{d^3y}{dx^3} + (1+x)^2\frac{d^2y}{dx^2} + 3(1+x)\frac{dy}{dx} - 8y = \frac{x}{(1+x)^{\frac{1}{2}}}.$$

Let $\dfrac{dx}{1+x} = dt$, and therefore $1 + x = \epsilon^t$. The transformed equation is

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 8y = \epsilon^{\frac{t}{2}} - \epsilon^{-\frac{t}{2}},$$

the integral of which is

$$y = \frac{8}{85}\,\frac{1}{(1+x)^{\frac{1}{2}}} - \frac{8}{51}(1+x)^{\frac{1}{2}} + C(1+x)^2$$
$$+ C_1 \cos\log(1+x)^2 + C_2 \sin\log(1+x)^2.$$

(8) Let $$x^3\frac{d^3y}{dx^3} - 3x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} - 8y = X.$$

When the independent variable is changed, the operating function is found to contain three equal factors, hence the integral is

$$y = x^2\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\,\frac{X}{x^2} + x^2\left\{C_0 + C_1\log x + C_2(\log x)^2\right\}.$$

In other cases the reduction may be made by artifices suggested by the form of the equation.

(9) Let $$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - a^2y = 0.$$

Now $$\frac{d^2(xy)}{dx^2} = x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}.$$

Therefore $$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} = \frac{1}{x}\frac{d^2(xy)}{dx^2}.$$

The given equation may therefore be put under the form

$$\frac{d^2(xy)}{dx^2} - a^2(xy) = 0;$$

which is a linear equation with constant coefficients. The integral of this is evidently

$$xy = C\epsilon^{ax} + C_1\epsilon^{-ax};$$

and therefore $$y = \frac{1}{x}(C\epsilon^{ax} + C_1\epsilon^{-ax})$$

is the integral of the given equation.

(10) Let $$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \left(n^2 - \frac{2}{x^2}\right)y = 0.$$

This may be put under the form

$$\frac{d}{dx}\left(\frac{d}{dx} + \frac{2}{x}\right)y + n^2y = 0.$$

Integrating with respect to x, that is, operating on both sides of the equation with $\left(\frac{d}{dx}\right)^{-1}$, we have

$$\left(\frac{d}{dx} + \frac{2}{x}\right)y + n^2\left(\frac{d}{dx}\right)^{-1}y = C,$$

C being an arbitrary constant.

Multiplying by x,

$$\left(x\frac{d}{dx} + 2\right)y + n^2x\left(\frac{d}{dx}\right)^{-1}y = Cx.$$

Now $$\left(x\frac{d}{dx} + 2\right)y = \left(\frac{d}{dx}\right)^2\left\{x\left(\frac{d}{dx}\right)^{-1}y\right\}.$$

If therefore we put $z = x\left(\frac{d}{dx}\right)^{-1}y$, the equation becomes

$$\frac{d^2z}{dx^2} + n^2z = Cx;$$

the integral of which is

$$z = \frac{C}{n^2} x + A \cos (nx + a),$$

A and a being arbitrary constants. Therefore as

$$y = \frac{d}{dx} \left(\frac{z}{x}\right), \text{ we find}$$

$$y = -\frac{A}{x^2} \cos (nx + a) - \frac{nA}{x} \sin (nx + a).$$

The integrals of other equations with variable coefficients will be found in the following chapter on Integration by Series.

After all however, when these equations are of the second or higher orders, the number of cases in which they are integrable is very limited, and there seems to be no great prospect of the number being much increased. A little consideration will point out the reason of this. When we speak of an equation being integrable, we mean that the dependent variable can be expressed in terms of the dependent variable by means of a finite series of functions of that quantity, the forms of such functions being limited to those known as algebraical and transcendental. Now it has been seen that the simplest forms of differential equations involve the highest transcendants which we recognize as known functions, such as ϵ^{ax} or $\cos nx$, and it is to be expected that when the equations become more complicated their integrals must involve higher transcendants to which we have not affixed particular names, and which we do not look on as known forms. This indeed is found to be the case, as for example in the equation

$$x \frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0,$$

which in its integral involves the transcendant

$$\psi (x) = 1 - \frac{x}{1^2} + \frac{x^2}{1^2 . 2^2} - \frac{x^3}{1^2 . 2^2 . 3^2} + \frac{x^4}{1^2 . 2^2 . 3^2 . 4^2} - \&c.$$

It would appear then that before we are able to make any farther progress in the solution of differential equations we must create new transcendants in the same way as the ordinary transcendants ϵ^x, $\cos x$, $\log x$, &c. have been created, we must study their properties, and endeavour to express the integrals of differential equations by means of them. The first part of this task has for some time past occupied the attention of mathematicians, and great progress has been made in it though much still remains to be done. The second part has also been the object of study, though not to the same extent as the other, and several mathematicians have applied themselves with success to the expression of the integrals of differential equations by means of definite integrals which are the representatives of new transcendants. Thus for instance in the case cited above, the transcendant

$$1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \frac{x^4}{1^2.2^2.3^2.4^2} - \&c. = \frac{1}{\pi}\int_0^\pi d\theta \cos(2\sin\theta\, x^{\frac{1}{2}})..$$

Examples of such integrals will be found in Crelle's *Journal*, Vol. x. p. 92; Vol. XII. p. 144; Vol. XVII. p. 363.

As it appears then that the number of linear differential equations, which are integrable by means by the ordinary transcendants, is not very great, it becomes a matter of some importance to enquire under what circumstances they are so integrable, and to classify them accordingly. This enquiry has been undertaken by M. Liouville, whose researches on the subject will be found in the *Jour. de l'Ecole Polyt.* XXII^e. Cahier, p. 149, and in the *Mémoires des Sav. Etran.* Vol. v. p. 108.

There are however some general properties of these equations which may be studied without a knowledge of their complete solution, and which are of importance in the absence of more direct ways of attacking them. Such is the theorem of Lagrange, that if we have a linear equation of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + P_2 \frac{d^{n-2}y}{dx^{n-2}} + \&c. + P_n y = \phi(x) \ldots (1),$$

and if we know r values of y which satisfy it, when the second side vanishes, the equation can always be reduced to the integration of a linear equation of the order $n - r$. The following demonstration is due to Libri.*

Let y_1 be the value of y which satisfies the given equation, when the second side becomes zero.

Assume $y = y_1 \int z\,dx$, z being a new variable. Then, observing that by the theorem of Leibnitz,

$$d^p (uv) = v\,d^p u + p\,dv\,d^{p-1} u + \frac{p(p-1)}{1.2} d^2 v\,d^{p-2} u + \&c.$$

the given equation takes the form

$$y_1 \frac{d^{n-1} z}{dx^{n-1}} + \left(n \frac{dy_1}{dx} + P_1 y_1\right) \frac{d^{n-2} z}{dx^{n-2}} + \&c.$$

$$+ \left(\frac{d^n y_1}{dx^n} + P_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \&c. + P_n y_1\right) \int z\,dx = \phi(x).$$

Now since y_1 satisfies the equation

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^n} + \&c. + P_n y = 0, \; \ldots\ldots \; (2)$$

the term involving $\int z\,dx$ disappears, and on dividing by y_1 we have an equation of the form

$$\frac{d^{n-1} z}{dx^{n-1}} + Q_1 \frac{d^{n-2} z}{dx^{n-2}} + \&c. + Q_{n-1} z = \frac{\phi(x)}{y_1}, \; \ldots \; (3)$$

and the equation is thus reduced to one of an order lower by unity. Again, if y_2 be another value of y which satisfies equation (2), and since $z = \frac{d}{dx}\left(\frac{y}{y_1}\right)$, we have $z_1 = \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$ as a particular integral of

$$\frac{d^{n-1} z}{dx^{n-1}} + Q_1 \frac{d^{n-2} z}{dx^{n-2}} + \&c. + Q_{n-1} z = 0.$$

If therefore we assume $z = z_1 \int z'\,dx = \frac{d}{dx}\left(\frac{y_2}{y_1}\right) \int z'\,dx$, we shall be able as before to reduce the equation (3) to

$$\frac{d^{n-2} z'}{dx^{n-2}} + Q'_1 \frac{d^{n-3} z'}{dx^{n-3}} + \&c. + Q'_{n-2} z' = \frac{\phi(x)}{y_1 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)},$$

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* Crelle's *Journal*, Vol. x. p. 186.

which is of an order inferior to (1) by two unities. Proceeding in this manner we can reduce the equation to one of the order $n - r$.

In the same memoir M. Libri has shewn that linear differential equations possess various properties analogous to those of ordinary equations. Thus, for instance, we can always cause to disappear the $(r-1)^{\text{th}}$ term of such an equation, by the aid of the solution of an equation of the r^{th} order. Let the equation be

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \&c. + P_r \frac{d^{n-r}y}{dx^{n-r}} + \&c. + P_n = 0,$$

and let us assume $y = zu$; it then becomes

$$u\frac{d^n z}{dx^n} + \left(n\frac{du}{dx} + P_1 u\right)\frac{d^{n-1}z}{dx^{n-1}} + \&c.$$

$$+ \frac{n(n-1)\dots\dots(n-r+1)}{1.2\dots\dots r}\left\{\frac{d^r u}{dx^r} + \frac{r}{n}P_1\frac{d^{r-1}u}{dx^{r-1}}\right.$$

$$\left. + \frac{r(r-1)}{n(n-1)}P_2\frac{d^{r-2}u}{dx^{r-2}} + \&c.\right\}\frac{d^{n-r}z}{dx^{n-r}} + \&c. + \&c. = 0.$$

The $(r+1)^{\text{th}}$ term of this transformed equation will disappear if

$$\frac{d^r u}{dx^r} + \frac{r}{n}P_1\frac{d^{r-1}u}{dx^{r-1}} + \frac{r(r-1)}{n(n-1)}\frac{d^{r-1}u}{dx^{r-1}} + \&c. = 0,$$

which is a linear equation of the r^{th} order. Since

$$n\frac{du}{dx} + P_1 u = 0,$$

can always be solved, it appears that we can always make the second term of a linear equation disappear.

Some equations which are not linear may be reduced to that form by a change of the variable.

Let $dy + Py\,dx = Xy^{n+1}dx$;

assume $y^n = \frac{1}{u}$, when the equation becomes

$$du - nPu\,dx = -nX\,dx,$$

which is linear with respect to u.

See Jac. Bernoulli, *Opera*, pp. 663 and 731.

(11) Let $dy + y\,dx = xy^3\,dx$.

The equation in u is

$$du - 2u\,dx = -2x\,dx;$$

from which $u = \frac{1}{y^2} = x + \frac{1}{2} + Ce^{2x}$.

(12) Let $dy + \frac{xy\,dx}{1 - x^2} = xy^{\frac{1}{2}}\,dx$.

The integral is

$$y^{\frac{1}{2}} = -\frac{1 - x^2}{3} + C(1 - x^2)^{\frac{1}{4}}.$$

(13) Let $ay\,dy - by^2\,dx = cx\,dx$.

Assuming $y^2 = u$ this becomes

$$a\,du - 2bu\,dx = 2cx\,dx,$$

which is linear in u. The integral is

$$u = y^2 = -\frac{c}{b}x - \frac{ac}{2b^2} + Ce^{\frac{2b}{a}x},$$

(14) Let $xy^2\,dy + y^3\,dx = \frac{a^3\,dx}{x}$.

By assuming $y^3 = u$, we find

$$u = y^3 = \frac{3a^3}{2x} + \frac{C}{x^3}.$$

(15) Let $y\,dy - \frac{ay^2}{x^2}\,dx = \frac{b}{x^3}$.

The integral found by assuming $y^2 = u$ is

$$y^2 = e^{-\frac{2a}{x}} - \frac{b}{ax} + \frac{b}{2a^2}.$$

(16) Let $dx - xy\,dy = x^2y^3\,dy$.

Putting $x = \frac{1}{v}$, this becomes

$$\frac{dv}{dy} + yv = -y^3,$$

which is linear with respect to v. The integral is

$$\frac{1}{x} = 2 - y^2 + C\epsilon^{-\frac{y^2}{2}}.$$

SECT. 3. *Equations integrable by separating the variables.*

I. Homogeneous equations of the first order and degree can always be integrated by means of the separation of the variables. If the two variables be x and y, assume

$$\frac{y}{x} = z, \quad \text{or} \quad \frac{x}{y} = z,$$

and by means of one of these equations and its differential eliminate one of the variables and its differential from the given equation. The resulting equation involving z and the other variable always admits of the variables being separated.

This method of integrating homogeneous differential equations of the first order was first given by John Bernoulli. See the *Comm. Epis.* of Leibnitz and Bernoulli, Vol. I. p. 7.

Ex. (1) Let the equation be

$$xdx + ydy = mydx.$$

Assuming $\frac{y}{x} = z$, the transformed equation is

$$\frac{dx}{x} + \frac{zdz}{1 - mz + z^2} = 0,$$

$$\text{or} \quad \frac{dx}{x} + \frac{1}{2}\,\frac{2z - mdz}{1 - mz + z^2} + \frac{1}{2}\,\frac{mdz}{1 - mz + z^2} = 0;$$

the integral of which is

$$\log x + \frac{1}{2}\log(1 - mz + z^2) + \frac{m}{2}\int\frac{dz}{1 - mz + z^2} = C.$$

If $m > 2$, the denominator of the part under the sign of integration is of the form $(z - a)\left(z - \frac{1}{a}\right)$, and therefore

$$\int\frac{mdz}{1 - mz + z^2} = \frac{a^2 + 1}{a^2 - 1}\log\left(\frac{z - a}{z - \frac{1}{a}}\right),$$

and $\log x + \frac{1}{2}\log(1 - mz + z^2) + \frac{1}{2}\frac{a^2+1}{a^2-1}\log\left(\frac{z-a}{z-\frac{1}{a}}\right) = C.$

Substituting for z its value $\frac{y}{x}$, we have

$$\log(x^2 - mxy + y^2)^{\frac{1}{2}} + \frac{a^2+1}{2(a^2-1)}\log\left(\frac{ay-a^2x}{ay-x}\right) = C.$$

Let $m < 2$, so that we may assume $m = 2\cos\alpha$. Then

$$\int\frac{dz}{1-2\cos\alpha . z + z^2} = \frac{1}{\sin\alpha}\tan^{-1}\left(\frac{z\sin\alpha}{1-z\cos\alpha}\right);$$

and therefore the integral of the equation is

$$\log(x^2 - mxy + y^2)^{\frac{1}{2}} + \cot\alpha\tan^{-1}\left(\frac{y\sin\alpha}{x-y\cos\alpha}\right) = C.$$

Let $m = 2$, or $1 - mz + z^2 = (1-z)^2$. Then the integral of the equation becomes

$$\log(x-y) = C - \frac{x}{x-y},$$

$$\text{or}\quad x - y = C\epsilon^{-\frac{x}{x-y}}.$$

(2) Let $xdy - ydx = (x^2+y^2)^{\frac{1}{2}}dx$.

Making $y = xz$, this becomes

$$\frac{dx}{x} = \frac{dz}{(1+z^2)^{\frac{1}{2}}},$$

$$\text{whence}\quad x = C\left\{z + (1+z^2)^{\frac{1}{2}}\right\},$$

$$\text{from which}\quad x^2 = 2Cy + C^2.$$

(3) Let $(x^2y + y^3)\,dx = 3xy^2dy$.

Assuming $y = xz$, we find

$$\frac{dx}{x} = \frac{3zdz}{1-2z^2},$$

an equation which is easily integrable, since the second side is a rational fraction. The final integral may be put under the form $\quad (x^2 - 2y^2)^3 = Cx^2.$

(4) Let $y^2dx + (xy + x^2)\,dy = 0.$

Assume $x = yz$, when the transformed equation becomes

$$\frac{dy}{y} + \frac{dz}{2z + z^2},$$

and the final integral is $y = C\left(\frac{2y + x}{x}\right)^{\frac{1}{2}}.$

(5) Let $xdx + ydy = m\,(xdy - ydx).$

Assuming $y = xz$, we find the integral to be

$$\log\,(x^2 + y^2)^{\frac{1}{2}} = m\tan^{-1}\frac{y}{x} + C.$$

(6) Let $y^2dx + x^2dy = xydy.$

Assuming $x = yz$, we find as the integral $y = C\epsilon^{\frac{y}{x}}.$

(7) Let $y^3dy + 3y^2x\,dx + 2x^3dx = 0.$

The integral of this is $y^2 + 2x^2 = C\,(x^2 + y^2)^{\frac{1}{2}}.$

(8) Let $x^2ydx - y^3dy = x^3dy.$

The integral is $y = C\epsilon^{\frac{x^3}{3y^3}}.$

(9) Let $xydy - y^2dx = (x + y)^2\,\epsilon^{-\frac{y}{x}}\,dx.$

The transformed equation is

$$\frac{dx}{x} = \frac{z\epsilon^z dz}{(1 + z)^2},$$

and the integral is

$$(x + y)\log\frac{x}{c} = x\epsilon^{\frac{y}{x}}.$$

(10) Let the equation be

$$x^3dy - x^2y\,dx + y^3dx - xy^2dy = 0.$$

In this case the transformed equation is reduced to

$$x\,(1 - z^2)\,dz = 0\,;$$

which may be satisfied by

$$x = 0, \quad 1 - z^2 = 0, \quad \text{or } dz = 0.$$

This last is the only differential equation, and therefore is the solution of the equation. It gives as the integral

$$z = c, \quad \text{or} \quad y = cx.$$

The other two solutions correspond to particular values of the arbitrary constant. The first or $x = 0$ gives $c = \infty$, the second or $z^2 = 1$ gives $C = \pm 1$.

II. Equations in which the variables can be separated by particular assumptions.

(11) Let $(mx + ny + p)\,dx + (ax + by + c)\,dy = 0.$

Assume $ax + by + c = z, \quad mx + ny + p = u;$

whence $adx + bdy = dz, \quad mdx + ndy = du,$

and therefore

$$dy = \frac{mdz - adu}{mb - na}, \quad dx = \frac{bdu - ndz}{mb - na},$$

by means of which the proposed equation becomes

$$(mz - nu)\,dz + (bu - az)\,du = 0,$$

which is a homogeneous equation integrable by the usual assumption.

If $\frac{m}{n} = \frac{a}{b}$ this method fails, but the given equation is then easily integrable: for eliminating m it becomes

$$b\,(cdy + pdx) + (ax + by)\,(bdy + ndx) = 0;$$

and by assuming $ax + by = z$ whence $bdy = dz - adx$, the equation becomes

$$\{ac - bp + (a - n)\,z\}\,dx = (c + z)\,dz,$$

in which the variables are separated.

Euler, *Calc. Integ.* Vol. I. p. 261.

(12) Let $dy = (a + bx + cy)\,dx.$

By assuming $bx + cy = z$ we find the integral to be

$$b + c\,(a + bx + cy) = C\epsilon^{cx}.$$

Euler, *Ib.* p. 262.

(13) Let $dy + b^2y^2dx = a^2x^m dx$.

Assume $y = z^r$, by which the equation becomes

$$rz^{r-1}dz + b^2z^{2r}dx = a^2x^m dx.$$

In order that this may be homogeneous we must have

$$r - 1 = 2r = m;$$

whence $r = -1$, $m = -2$, so that the transformed equation is

$$-\frac{dz}{z^2} + b^2\frac{dx}{z^2} = \frac{a^2dx}{x^2},$$

a homogeneous equation in which the variables are separable.

This equation was first considered by Riccati in the *Acta Eruditorum*, Sup. VIII. p. 66, and it usually bears his name. It may be converted into a linear equation by assuming

$$y = \frac{1}{b^2z}\frac{dz}{dx},$$

when it becomes

$$\frac{d^2z}{dx^2} - a^2b^2x^m z = 0.$$

(14) If in the equation of Riccati $m = 0$, the variables are immediately separable. It becomes then

$$dy + b^2y^2\,dx = a^2dx \quad \text{or} \quad dx = \frac{dy}{a^2 - b^2y^2},$$

the integral of which is

$$\frac{a + by}{a - by} = Ce^{2abx}.$$

The assumption $y = z^k$ is not the only one which renders the equation of Riccati integrable. If we assume

$$y = Ax^p + x^q z,$$

the equation becomes

$$x^q dz + (qx^{q-1} + 2b^2Ax^{p+q} + b^2x^{2q})\,z\,dx + (pAx^{p-1} + b^2A^2x^{2p})\,dx$$
$$= a^2x^m\,dx.$$

This will be reduced to an equation of three terms, if we have

$$p - 1 = 2p, \qquad q - 1 = p + q,$$
$$pA + b^2A^2 = 0, \qquad q + 2b^2A = 0.$$

The first and second conditions agree in giving $p = -1$, and from the second and third we find

$$A = \frac{1}{b^2}, \quad q = -2;$$

so that the assumption is

$$y = \frac{1}{b^2 x} + \frac{z}{x^2},$$

and the equation is then reduced to

$$dz + b^2 z^2 \frac{dx}{x^2} = a^2 x^{m+2} dx.$$

(15) If $m = -4$, the equation becomes

$$dz + b^2 z^2 \frac{dx}{x^2} = \frac{a^2 dx}{x^2},$$

in which the variables are separated. The integral is

$$\frac{ab + x - b^2 x^2 y}{ab - x + b^2 x^2 y} = C\epsilon^{\frac{2ab}{x}}.$$

If in the equation

$$dz + b^2 z^2 \frac{dx}{x^2} = a^2 x^{m+2} dx \text{ we assume } z = \frac{1}{u},$$

we have
$$du + a^2 u^2 x^{m+2} dx = \frac{b^2 dx}{x^2};$$

and in this equation making $(m+3)\, x^{m+2} dx = dv$, and for shortness putting

$$\frac{a^2}{m+3} = \beta^2, \quad \frac{b^2}{m+3} = \alpha^2, \quad \frac{m+4}{m+3} = -n,$$

it is reduced to

$$du + \beta^2 u^2 dv = \alpha^2 v^n dv,$$

which is similar to the proposed equation, and is therefore integrable if $n = -4$, or $m = -\frac{8}{3}$. If n be not equal to -4, we may transform this equation by the same assumptions as before, when we shall obtain an equation of the form
$$du' + \beta'^2 u'^2 dv' = \alpha'^2 v'^{n'} dv',$$

which is integrable if $n' = -4$, furnishing a corresponding value for m. In this way we may proceed, continually transforming the equation and finding values of m which render Riccati's equation integrable. It will be found that these values are included in the formula

$$m = -\frac{4r}{2r-1},$$

r being an integer.

Another series of values for m may be found by making $y = \frac{1}{u}$ in the original equation, when it becomes

$$du + a^2u^2x^m dx = b^2 dx;$$

and this being transformed by the assumptions

$$x^{m+1} = v, \quad \frac{a^2}{m+1} = \beta^2, \quad \frac{b^2}{m+1} = \alpha^2, \quad \frac{m}{m+1} = -n,$$

we find $$du + \beta^2u^2 dv = \alpha^2 v^n dv,$$

which is similar to the proposed equation and integrable if n be of the form $-\frac{4r}{2r-1}$, that is, if

$$\frac{m}{m+1} = \frac{4r}{2r-1}, \text{ or } m = -\frac{4r}{2r+1}.$$

Hence all the values of m are included in the formula

$$m = -\frac{4r}{2r \pm 1}.$$

(16) Let $dy + y^2 dx = \frac{dx}{x^{\frac{4}{3}}}$.

Then $$\frac{y(x^{\frac{1}{3}} + 3x^{\frac{2}{3}}) + 3}{y(x^{\frac{1}{3}} - 3x^{\frac{2}{3}}) + 3} = C\epsilon^{6x^{\frac{1}{3}}}.$$

(17) Let $dy - y^2 dx = \frac{2dx}{x^{\frac{8}{3}}}$.

Then $$\frac{x^{\frac{2}{3}} + yx^{\frac{5}{3}} - 6}{3.2^{\frac{1}{2}}x^{\frac{1}{3}}(1+xy)} = \tan\left(\frac{3.2^{\frac{1}{2}}}{x^{\frac{1}{3}}} + C\right).$$

(18) The equation

$$dy + ay^n x^p dx + bx^m y^q dx = 0$$

can be made homogeneous if

$$(p+1)(1-q)=(m+1)(1-n),$$

by the assumption

$$y=x^{\frac{p+1}{1-n}} \text{ or } =x^{\frac{m+1}{1-q}}.$$

There is an exception to this if $n=1$ and $q=1$; but in this case the equation becomes

$$dy+y(ax^p+bx^m)\,dx=0,$$

in which the variables are already separated.

(19) Let $aydx+bxdy+x^m y^n(cydx+exdy)=0$; dividing by xy we have

$$a\frac{dx}{x}+b\frac{dy}{y}+x^m y^n\left(c\frac{dx}{x}+e\frac{dy}{y}\right)=0.$$

From this it appears that the assumptions

$$x^a y^b=u, \quad x^c y^e=v$$

will simplify the equation. It becomes after these substitutions

$$\frac{du}{u}+u^\alpha v^\beta\frac{dv}{v}=0,$$

$$\text{where } \alpha=\frac{me-nc}{ae-bc}, \; \beta=\frac{na-mb}{ae-bc}.$$

The integral of this is evidently

$$-\frac{u^{-\alpha}}{\alpha}+\frac{v^\beta}{\beta}=C.$$

If $\alpha=0$ and $\beta=0$, i.e. if $\frac{m}{n}=\frac{a}{b}=\frac{c}{e}$, the integral takes the form

$$\log u+\log v=C;$$

$$\text{or} \qquad uv=C;$$

$$\text{or} \qquad x^{a+c}y^{b+e}=C.$$

(20) Let $(x+y)^2\,dy=a^2\,dx$.

Assume $$x + y = u,$$

whence $$dy = \frac{a^2\,du}{a^2 + u^2},$$

in which the variables are separated.' The integral is

$$y + x = a \tan \frac{y + c}{a}.$$

(21) Let $(y - x)(1 + x^2)^{\frac{1}{2}}\,dy = n(1 + y^2)^{\frac{3}{2}}\,dx.$

To separate the variables assume

$$y = \frac{x - u}{1 + xu},$$

when the equation becomes

$$\frac{dx}{1 + x^2} = \frac{u\,du}{(1 + u^2)\,\{u + n(1 + u^2)^{\frac{1}{2}}\}}.$$

To integrate this put $1 + u^2 = t^2$, which gives

$$\frac{dx}{1 + x^2} = \frac{dt}{t\,\{nt + (t^2 - 1)^{\frac{1}{2}}\}};$$

and again putting $t = \dfrac{1 + s^2}{2s}$, we find

$$\frac{dx}{1 + x^2} = \frac{2\,ds}{1 + s^2} - \frac{2n\,ds}{(n + 1) + (n - 1)\,s^2},$$

which is easily integrable.

Euler, *Calo. Integ.* Vol. I. p. 270.

Sect. 4. *Equations which involve* y *and its differentials in powers and products.*

I. Equations of the form

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + \&c. + P_n = 0,$$

are to be resolved (when possible) into the simple factors

$$\left(\frac{dy}{dx} - u_1\right)\left(\frac{dy}{dx} - u_2\right)\ldots\ldots\left(\frac{dy}{dx} - u_n\right) = 0;$$

and each of these is to be integrated separately. Any one of these integrals, or the product of any number of them, will be an integral of the proposed equation.

(1) Let $\left(\frac{dy}{dx}\right)^2 - a^2 = 0$.

Here $$\frac{dy}{dx} - a = 0, \qquad \frac{dy}{dx} + a = 0;$$

therefore $$y = ax + c, \qquad y = -ax + c_1,$$

are both integrals: also
$$(y - ax - c)(y + ax - c_1) = 0.$$

If we suppose c and c_1 to be the same, this may be put under the form
$$(y - c)^2 = a^2x^2.$$

(2) Let $y^2\left(\frac{dy}{dx}\right)^2 - 4a^2 = 0$.

The integrals are
$$y^2 = 4ax + c, \qquad y^2 = -4ax + c_1,$$

and $$(y^2 - 4ax - c)(y^2 + 4ax - c_1) = 0;$$

or $$(y^2 - c)^2 = 16a^2x^2, \quad \text{when } c_1 = c.$$

(3) Let $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$.

The integrals are
$$(x^2 + y^2)^{\frac{1}{2}} = x + c, \qquad (x^2 + y^2)^{\frac{1}{2}} = -x + c_1,$$

and $$\left\{(x^2 + y^2)^{\frac{1}{2}} - x - c\right\}\left\{(x^2 + y^2)^{\frac{1}{2}} + x - c_1\right\} = 0;$$

or $$y^2 = 2cx + c^2, \quad \text{when } c_1 = c.$$

(4) Let
$$\left(\frac{dy}{dx}\right)^3 - (x^2 + xy + y^2)\left(\frac{dy}{dx}\right)^2 + (x^3y + x^2y^2 + xy^3)\frac{dy}{dx} - x^3y^3 = 0.$$

The factors in this case are
$$\left(\frac{dy}{dx} - x^2\right)\left(\frac{dy}{dx} - xy\right)\left(\frac{dy}{dx} - y^2\right) = 0;$$

and the integrals are
$$y = \frac{x^3}{3} + c, \qquad y = \epsilon^{\frac{x^2}{2}} + c_1, \qquad y = -\frac{1}{x} + c_2,$$

and $$\left(y - \frac{x^3}{3} - c\right)\left(y - \epsilon^{\frac{x^2}{2}} - c_1\right)\left(y + \frac{1}{x} - c_2\right) = 0.$$

(5) Let

$$(a^2 - x^2)\left(\frac{dy}{dx}\right)^3 + bx(a^2 - x^2)\left(\frac{dy}{dx}\right)^2 - \frac{dy}{dx} - bx = 0.$$

The integrals are

$$y = c + \sin^{-1}\frac{x}{a}, \qquad y = c_1 - \sin^{-1}\frac{x}{a}, \qquad y = c_2 - \frac{bx^2}{2};$$

and $$(y - c)^3 = \frac{bx^2}{2}\left(\sin^{-1}\frac{x}{a}\right)^2 \quad \text{if} \quad c_2 = c_1 = c.$$

(6) Let $$\left(1 - y^2 - \frac{y^4}{x^2}\right)\left(\frac{dy}{dx}\right)^2 - \frac{2y}{x}\frac{dy}{dx} + \frac{y^2}{x^2} = 0;$$

or $$\left(\frac{dy}{dx}\right)^2 - \frac{2y}{x}\frac{dy}{dx} + \frac{y^2}{x^2} = \left(y^2 + \frac{y^4}{x^2}\right)\left(\frac{dy}{dx}\right)^2.$$

Extracting the square root on both sides and multiplying by x,

$$x\frac{dy}{dx} - y = \pm y(x^2 + y^2)^{\frac{1}{2}}\frac{dy}{dx};$$

whence $$y = x\cot\left(C \pm \frac{y^2}{2}\right).$$

II. If the equation be of the first order and homogeneous in x and y, and if we assume $y = ux$ or $x = uy$ we shall obtain by the elimination of the variables an equation between u and $\frac{dy}{dx}$, which combined with the differential of ux or uy will give us the means of finding the relation between x and y.

(7) Let $$y - x\frac{dy}{dx} = nx\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

Put $\frac{dy}{dx} = p$, and $y = ux$, then

$$u = p + n(1 + p^2)^{\frac{1}{2}};$$

but $$dy = pdx = udx + xdu;$$

therefore $$\frac{dx}{x} = \frac{du}{p - u}.$$

But $$du = dp + \frac{n\,p\,dp}{(1+p^2)^{\frac{1}{2}}},$$

$$\text{and} \quad p - u = -n\,(1+p^2)^{\frac{1}{2}};$$

$$\text{therefore} \quad \frac{dx}{x} = -\frac{dp}{n\,(1+p^2)^{\frac{1}{2}}} - \frac{p\,dp}{1+p^2};$$

$$\text{and} \quad \log x = -\frac{1}{n}\log\{p + (1+p^2)^{\frac{1}{2}}\} - \tfrac{1}{2}\log(1+p^2) + C;$$

whence if $C = \log a$,

$$x = \frac{a}{(1+p^2)^{\frac{1}{2}}}\{(1+p^2)^{\frac{1}{2}} - p\}^{\frac{1}{n}},$$

$$\text{and} \quad y = \frac{a\,\{p + n\,(1+p^2)^{\frac{1}{2}}\}}{(1+p^2)^{\frac{1}{2}}}\{(1+p^2)^{\frac{1}{2}} - p\}^{\frac{1}{n}}.$$

whence we find

$$\frac{\{y^2 + (1-n^2)\,x^2\}^{\frac{1}{2}} - ny}{1-n^2} = a\left[\frac{\{y^2 + (1-n^2)\,x^2\}^{\frac{1}{2}} - y}{(1-n)\,x}\right]^{\frac{1}{n}}.$$

(8) Let $$y\frac{dy}{dx} + nx = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}(y^2 + nx^2)^{\frac{1}{2}}.$$

The integral is

$$\left(\frac{x}{c}\right)^{\left(\frac{n-1}{n}\right)^{\frac{1}{2}}} = \frac{y + (y^2 + nx^2)^{\frac{1}{2}}}{x}.$$

(9) Let $$y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = n\left(x + y\frac{dy}{dx}\right).$$

This equation is homogeneous in x and y, and may be treated like the preceding examples, but it is more convenient to proceed as follows. Square both sides, and solve the equation with respect to $y\dfrac{dy}{dx}$, which gives

$$y\frac{dy}{dx} + \frac{n^2 x}{n^2 - 1} = \pm\frac{\{(n^2-1)\,y^2 + n^2x^2\}^{\frac{1}{2}}}{n^2-1}.$$

Whence, dividing by the second side of the equation and integrating,

$$\{(n^2-1)\,y^2 + n^2x^2\}^{\frac{1}{2}} = \pm x + C.$$

III. Equations integrable by Differentiation.

If $y = xp + f(p)$ $\left(\text{where } p = \dfrac{dy}{dx}\right)$,

we have, on differentiating, $0 = \{x + f'(p)\}\, dp$.

This is satisfied by $dp = 0$, or $y = Cx + C'$, where $C' = f(C)$. The singular solution is found by eliminating p between the given equation and $x + f'(p) = 0$.

This equation is known by the name of *Clairaut's* form, having been first integrated by him. See *Mémoires de l'Académie des Sciences*, 1734, p. 196.

(10) Let $y = px + n(1 + p^2)^{\frac{1}{2}}$.

The general integral is $y = Cx + n(1 + C^2)^{\frac{1}{2}}$;
the singular solution is $x^2 + y^2 = n^2$.

(11) Let $y = px + p - p^2$.

The general integral is $y = C(x + 1 - C)$;
the singular solution is $4y = (1 + x)^2$.

(12) Let $y - px = a(1 - p^3)^{\frac{1}{3}}$.

The general integral is $y = Cx + a(1 - C^3)^{\frac{1}{3}}$;
the singular solution is $y^{\frac{3}{2}} - x^{\frac{3}{2}} = a^{\frac{3}{2}}$.

(13) Let $y = px - \dfrac{ap}{(1 + p^2)^{\frac{1}{2}}}$.

The general integral is $y = C\left\{x - \dfrac{a}{(1 + C^2)^{\frac{1}{2}}}\right\}$;
the singular solution is $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$.

Sometimes an equation which is not of Clairaut's form may be reduced to it by being multiplied by a factor.

(14) Let $ay\left(\dfrac{dy}{dx}\right)^2 + (2x - b)\dfrac{dy}{dx} - y = 0$.

Multiply by $4y$, and let $y^2 = u$, and $2y\,dx = du$.

Then $$a\left(\frac{du}{dx}\right)^2 + (4x - 2b)\frac{du}{dx} - 4u = 0,$$

$$\text{or}\quad u = x\frac{du}{dx} - \left\{\frac{b}{2}\frac{du}{dx} - \frac{a}{4}\left(\frac{du}{dx}\right)^2\right\},$$

which is of Clairaut's form. The general integral is

$$u = y^2 = Cx - \left(\frac{b}{2}C - \frac{a}{4}C^2\right).$$

The singular solution is $4ay^2 + (2x - b) = 0$.

(15) Let $axy\left(\frac{dy}{dx}\right)^2 + (bx^2 - ay^2 - ab)\frac{dy}{dx} - bxy = 0$.

On multiplying by $4xy$, and taking x^2 and y^2 as the new variables, the equation becomes of Clairaut's form, and the integral is

$$y^2 = Cx^2 - \frac{abC}{b + aC}.$$

The singular solution is $ay^2 + b(x - a^{\frac{1}{2}})^2 =$ [illegible]

If $$y = Px + Q,$$

where P and Q are both functions of p, we have by differentiation

$$dy = pdx = Pdx + \left(x\frac{dP}{dp} + \frac{dQ}{dp}\right)dp,$$

whence $$(p - P)\,dx = \left(x\frac{dP}{dp} + \frac{dQ}{dp}\right)dp,$$

which being a linear equation in x may be integrated, so that we have x expressed in terms of p, and as $y = \int p\,dx$, we can eliminate p and so obtain a relation between x and y.

(16) Let $y = xp^2 + p^2$.

The integral is $y^{\frac{1}{2}} = (x + 1)^{\frac{1}{2}} + C$.

(17) Let $y = (1 + p)x + p^2$.

Then $y = 2(1 - p) + C\epsilon^{-p}$.

Substituting in this the value of p derived from the equation, we have the required integral.

(18) Let $y - 2px = a(1 + p^2)^{\frac{1}{2}}$.

We find $p^2x = -\frac{a}{2}[p(1 + p^2)^{\frac{1}{2}} - \log\{p + (1 + p^2)^{\frac{1}{2}}\}] + C$.

By eliminating p between this and the given equation, the integral is determined.

(19) Let $y = x\{p - (1+p^2)^{\frac{1}{2}}\}$.

In this case $Q = 0$, and we have to integrate

$$\frac{dx}{x} = \frac{(1+p^2)^{\frac{1}{2}} - p}{(1+p^2)^{\frac{1}{2}}}\,dp,$$

and then to eliminate p. The result is

$$x^2 + y^2 = 2cx.$$

IV. Homogeneous equations of the second order.

If an equation involve x, y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, and if we assume x and y to be both of one dimension, $\frac{dy}{dx}$ will be of 0 dimensions, and $\frac{d^2y}{dx^2}$ will be of -1 dimensions. The equation then is said to be homogeneous when, adopting this scale, the sum of the indices in each term is the same. To integrate an equation of this form, let $\frac{dy}{dx} = p$, $\frac{d^2y}{dx^2} = q$; then by assuming $y = ux$, $q = \frac{v}{x}$, the quantity x can be eliminated so as to give a relation between u, v, and p. But as $dy = p\,dx = u\,dx + x\,du$, we have

$$\frac{dx}{x} = \frac{du}{p-u},$$

and as $dp = q\,dx$, we have also $v\,dx = x\,dp$.

Whence $\qquad v\,du = (p-u)\,dp.$

From this v may be eliminated by means of the given equation, and we have a differential equation of the first order between p and u: by integrating this we obtain p in terms of u, and then x in terms of u from

$$\frac{dx}{x} = \frac{du}{p-u},$$

in which the variables are separated.

(20) Let $x^3 \dfrac{d^2 y}{dx^2} = \left(y - x\dfrac{dy}{dx}\right)^2$;

put $y = ux, \quad q = \dfrac{v}{x}$, then

$$v = (u - p)^2, \quad \text{and} \quad dp = (p - u)\,du.$$

This being a linear equation is easily integrated, and we find

$$p = u + 1 + C\epsilon^u.$$

Then $$\frac{dx}{x} = \frac{du}{1 + C\epsilon^u} = \frac{\epsilon^{-u}du}{C + \epsilon^{-u}},$$

and $$\log\frac{C'}{x} = \log(C + \epsilon^{-u}) \text{ or } \epsilon^{-u} = \frac{C' - Cx}{x};$$

whence $$y = x\log\left(\frac{C' - Cx}{x}\right),$$

which is the required integral.

(21) Let $x^2 \dfrac{dy}{dx}\dfrac{d^2 y}{dx^2} - x\dfrac{dy}{dx} + y = 0$.

The integral is

$$C'x^C = \frac{(C - 1)\,x^{\frac{1}{2}} + \{2y + (C^2 - 1)x\}^{\frac{1}{2}}}{(C + 1)\,x^{\frac{1}{2}} - \{2y + (C^2 - 1)x\}^{\frac{1}{2}}}.$$

There is also a singular solution $y = Cx$.

Sometimes an equation may be considered homogeneous by reckoning x as of one dimension, y of n dimensions, and consequently $\dfrac{dy}{dx}$ of $(n - 1)$ dimensions, and $\dfrac{d^2 y}{dx^2}$ of $(n - 2)$ dimensions. In such cases assume $y = x^n u$, $p = x^{n-1}t$, $q = x^{n-2}v$; then by steps similar to those in the last case we arrive at a differential equation of the first order, between t and u, which being integrated will enable us to determine the relation between x and y.

(22) Let $x^4 \dfrac{d^2 y}{dx^2} = (x^3 + 2xy)\dfrac{dy}{dx} - 4y^2$.

Assume $y = x^2 u, \quad p = xt, \quad q = v$. Then

$$v = t(1 + 2u) - 4u^2.$$

But we have

$$du\,(v-t) = dt\,(t-2u), \quad \text{and therefore}$$

$$2u\,du\,(t-2u) = dt\,(t-2u).$$

This is satisfied by

$$2u\,du = dt, \quad \text{or by} \quad t-2u = 0.$$

The first gives $u^2 + C = t$, and therefore

$$\frac{dx}{x} = \frac{du}{u^2-2u+C}.$$

When $C = 1$, this gives

$$x^2 = (x^2 - y)\log\frac{x}{a}.$$

When $C = 1 - n^2$, this gives

$$x^{2n} = a\frac{(n+1)\,x^2 - y}{(n-1)\,x^2 + y}.$$

When $C = 1 + n^2$, this gives

$$y = x^2\left\{1 + n\tan\left(n\log\frac{x}{a}\right)\right\}.$$

The other factor $t - 2u = 0$, gives

$$y = Cx^2,$$

as a singular solution.

If x be reckoned of 0 dimensions so that y, $\dfrac{dy}{dx}$, $\dfrac{d^2y}{dx^2}$ are of the same dimensions, a homogeneous equation may be integrated by assuming

$$p = uy, \qquad q = vy; \quad \text{whence as}$$

$$dy = uy\,dx \quad \text{and} \quad u\,dy + y\,du = vy\,dx,$$

$$\frac{dy}{y} = u\,dx \quad \text{and} \quad du + u^2dx = v\,dx.$$

From this last if we eliminate v by means of the given equation, we have to find u in terms of x, by integrating an equation of the first order, and then by means of $\frac{dy}{y} = u\,dx$, we can determine the relation between x and y.

(23) Let $y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = \frac{y\frac{dy}{dx}}{(a^2+x^2)^{\frac{1}{2}}}$.

From this we find

$$v = u^2 + \frac{u}{(a^2+x^2)^{\frac{1}{2}}} \quad \text{and} \quad \frac{du}{u} = \frac{dx}{(a^2+x^2)^{\frac{1}{2}}};$$

therefore $$u = C\left\{x + (a^2+x^2)^{\frac{1}{2}}\right\};$$

whence $\log(C'y) = Ca^2\log\left\{x+(a^2+x^2)^{\frac{1}{2}}\right\} + Cx\left\{x+(a^2+x^2)^{\frac{1}{2}}\right\}$.

(24) Let $xy\frac{d^2y}{dx^2} = y\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2 + \frac{nx\left(\frac{dy}{dx}\right)^2}{(a^2-x^2)^{\frac{1}{2}}}$.

The integral is

$$\frac{(a^2-x^2)^{\frac{1}{2}}}{n} = b\log\frac{c\left\{nb + (a^2-x^2)^{\frac{1}{2}}\right\}}{ny},$$

b and c being arbitrary constants.

V. Equations of the second order in which one or other of the variables is wanting.

If the deficient variable be the dependent variable y, by putting $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ we have an equation of the first order between p and x, by the integration of which we obtain p in terms of x, or x in terms of p; and then by means of the equation

$$y = \int p\,dx = xp - \int x\,dp,$$

we can find the relation between x and y.

(25) Let $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = \frac{a^2}{2x}\frac{d^2y}{dx^2}$;

putting $\frac{dy}{dx} = p$ and $\frac{d^2y}{dx^2} = \frac{dp}{dx}$ this becomes

$$(1 + p^2)^{\frac{3}{2}} = \frac{a^2}{2x}\frac{dp}{dx};$$

therefore $$\frac{dp}{(1+p^2)^{\frac{3}{2}}} = \frac{2x\,dx}{a^2},$$

whence $$\frac{p}{(1+p^2)^{\frac{1}{2}}} = \frac{x^2}{a^2} + C = \frac{x^2+ab}{a^2},$$

and $$y = \int \frac{(x^2+ab)\,dx}{[\{a^4-(x^2+ab)^2\}]^{\frac{1}{2}}}.$$

This is the equation to the elastic curve.

Jac. Bernoulli, *Opera*, p. 576.

(26) Let $(1+x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0.$

The integral is

$$C^2y = (1+C^2)\log(1+Cx) - Cx + C'.$$

(27) Let $1 + \left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} = a\dfrac{d^2y}{dx^2}\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$

The equation between p and x is

$$(1+p^2)^{\frac{1}{2}} + xp\frac{dp}{dx} = a\frac{dp}{dx}(1+p^2)^{\frac{1}{2}},$$

which is integrable when divided by $(1+p^2)^{\frac{1}{2}}$.

The complete integral is

$$y = (a^2+b^2-x^2)^{\frac{1}{2}} - b\log\frac{b+\{(a^2+b^2-x^2)\}^{\frac{1}{2}}}{c(x-a)},$$

where b and c are the arbitrary constants.

(28) Let $a^2\dfrac{d^2y}{dx^2}(a^2+x^2)^{\frac{1}{2}} + a^2\dfrac{dy}{dx} = x^2.$

The integral is

$$a^2y^2 = -\frac{x^3}{9} - \frac{2a^2x}{3} + \frac{2(a^2+x^2)^{\frac{3}{2}}}{9} - Cx^2$$
$$+ Cx(a^2+x^2)^{\frac{1}{2}} + a^2C\log\frac{x+(a^2+x^2)^{\frac{1}{2}}}{C'}.$$

(29) Let $(x+a)\dfrac{d^2y}{dx^2} + x\left(\dfrac{dy}{dx}\right)^2 = \dfrac{dy}{dx}.$

The equation

$$(x+a)\frac{dp}{dx} - p = -xp^2.$$

becomes linear by assuming $p = \frac{1}{v}$, and the complete integral is

$$y + c' = \log(x^2 - c^2) - \frac{a}{c}\log\left(\frac{x+c}{x-c}\right),$$

c and c' being arbitrary constants.

If the independent variable (x) be wanting, we put $\frac{d^2y}{dx^2} = \frac{dy}{dx}\cdot\frac{dp}{dy} = p\cdot\frac{dp}{dy}$, and then we have an equation between p and y from which by integration we find p in terms of y, or y in terms of p, and then x is known from the equation

$$dy = p\,dx.$$

(30) Let $\frac{d^2y}{dx^2}\left(y\frac{dy}{dx} + 1\right) = \frac{dy}{dx}\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}$.

From this $\quad \frac{dp}{dy}(py + 1) = 1 + p^2,$

and $\quad dy - \frac{p}{1+p^2}\,y\,dp = \frac{dp}{1+p^2},$

a linear equation in y, which being integrated gives

$$y = p + C(1+p^2)^{\frac{1}{2}},$$

$$x = \int\frac{dy}{p} = \log p + C\log\{p + (1+p^2)^{\frac{1}{2}}\} + C',$$

whence by eliminating p we obtain a relation between x and y.

(31) Let $\left\{y^2 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = ny\left\{2\left(\frac{dy}{dx}\right)^2 + y^2 - y\frac{d^2y}{dx^2}\right\}$.

The integral is

$$x = C + \left(\frac{2y-a}{a}\right)^{\frac{1}{2}} + \cos^{-1}\frac{y-a}{y},$$

where C and a are arbitrary constants.

(32) Let $y\dfrac{d^2y}{dx^2} + \left(\dfrac{dy}{dx}\right)^2 = 1$, or $yp\dfrac{dp}{dy} + p^2 = 1$.

Putting this under the form

$$p\left(y\frac{dp}{dy} + p\right) = 1,$$

multiplying by y and integrating, we have

$$(py)^2 = a^2 + y^2;$$

$$\text{whence}\quad x + c = (a^2 + y^2)^{\frac{1}{2}}.$$

(33) Let $\left(\dfrac{dy}{dx}\right)^2 - y\dfrac{d^2y}{dx^2} = n\left\{\left(\dfrac{dy}{dx}\right)^2 + a^2\left(\dfrac{d^2y}{dx^2}\right)^2\right\}^{\frac{1}{2}}$.

Putting $\dfrac{d^2y}{dx^2} = p\dfrac{dp}{dy}$, this becomes

$$p - y\frac{dp}{dy} = n\left\{1 + a^2\left(\frac{dp}{dy}\right)^2\right\}^{\frac{1}{2}},$$

which is of Clairaut's form. The general integral is therefore

$$p = Cy + n(1 + a^2C^2)^{\frac{1}{2}},$$

$$\text{whence}\quad Cx + C' = \log\left\{Cy + n(1 + a^2C^2)^{\frac{1}{2}}\right\}.$$

The singular solution is

$$p = \frac{(n^2a^2 - y^2)^{\frac{1}{2}}}{a};$$

$$\text{whence}\quad y = na\sin\frac{C + x}{a}.$$

The examples in this section are taken chiefly from Euler, *Calc. Integ.* Vol. I. Sec. III. and Vol. II. Sec. I. Cap. 2 and 3.

CHAPTER V.

INTEGRATION OF DIFFERENTIAL EQUATIONS BY SERIES.

The method employed for integrating Differential Equations by series, is to assume an expression for the dependent variable in terms of the independent variable with indeterminate coefficients and indices, and then to determine them by the condition of the given equation.

(1) Let $\dfrac{d^2y}{dx^2} + ax^ny = 0.$

Assume $y = x^\alpha (A + A_1x^{n+2} + A_2x^{2n+4} + A_3x^{3n+6} + \text{\&c.})$

Whence we find

$$\frac{d^2y}{dx^2} = \alpha(\alpha-1)Ax^{\alpha-2} + (\alpha+n+2)(\alpha+n+1)A_1x^{\alpha+n} + \text{\&c.}$$

and $ax^ny = \qquad aAx^{\alpha+n} + aA_1x^{\alpha+2n+2} + \text{\&c.}$

Substituting these values in the equation, and equating to zero the coefficients of the powers of x, we have

$$\alpha(\alpha-1)A = 0, \quad (\alpha+n+2)(\alpha+n+1)A_1 + aA = 0,$$
$$(\alpha+2n+4)(\alpha+2n+3)A_2 + aA_1 = 0, \text{ \&c.}$$

The first of these is satisfied either by $\alpha = 0$ or $\alpha = 1$. Taking $\alpha = 0$ and substituting it in the other equations, we find

$$A_1 = -\frac{aA}{(n+1)(n+2)}, \quad A_2 = \frac{a^2A}{1.2(n+1)(2n+3)(n+2)^2},$$
$$A_3 = -\frac{a^3A}{1.2.3(n+1)(2n+3)(3n+5)(n+2)^3} \text{ \&c. \&c.}$$

so that

$$y = A\left\{1 - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{a^2x^{2n+4}}{1.2(n+1)(2n+3)(n+2)^2} - \text{\&c.}\right\}$$

But as this contains only one arbitrary constant A, it is not the complete solution. Let us take $a = 1$ and call A', A_1', A_2', &c. the corresponding coefficients; we then find in the same way as before

$$y = A' \left\{x - \frac{ax^{n+3}}{(n+3)(n+2)} + \frac{a^2x^{2n+5}}{1.2(n+3)(2n+5)(n+2)^2} - \&c.\right\}$$

which is another incomplete integral with one arbitrary constant. The sum of these two series is the complete integral of the equation.

When $n = -2$ both the series fail, as the denominators are then infinite: but the true integral is easily found.

For if $$\frac{d^2y}{dx^2} + \frac{ay}{x^2} = 0,$$

and we assume $y = Ax^a$, we have

$$a(a-1) + a = 0.$$

This is a quadratic equation, which gives two values for a. If these be a_1, a_2 the integral is

$$y = A_1x^{a_1} + A_2x^{a_2}.$$

The first of the preceding series will fail when $n = -\dfrac{2r-1}{r}$, and the second when $n = -\dfrac{(2r+1)}{r}$, r being any whole number: the complete integral may however be found by the following process. Assume

$$y = u + v \log cx,$$

where v is the particular integral furnished by the series which does not fail. On substituting this value of y in the original equation we obtain the system of equations

$$\frac{d^2v}{dx^2} + ax^nv = 0,$$

$$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{dv}{dx} - \frac{v}{x^2} + aux^n = 0;$$

the second of which serves to determine u. Euler, *Calc. Integ.* Vol. II. Chap. VII.

(2) Let $\dfrac{d^2y}{dx^2}+\dfrac{ay}{x}=0.$

Then $v=A\left\{x-\dfrac{ax^2}{1.2}+\dfrac{a^2x^3}{1.2^2.3}-\dfrac{a^3x^4}{1.2^2.3^2.4}+\&c.\right\}$

The equation to determine u is

$$\frac{d^2u}{dx^2}+\frac{2}{x}\frac{dv}{dx}-\frac{v}{x^2}+\frac{au}{x}=0.$$

Assume $\quad u=B+B_1x+B_2x^2+B_3x^3+\&c.$

Then we find $B=-\dfrac{A}{a}$; B_1 is left undetermined and

$$B_2=\frac{3aA}{1^3.2^2}-\frac{aB_1}{1.2},$$

$$B_3=\frac{-5a^2A}{1^2.2^3.3^2}-\frac{a^2A}{1^3.2^3}+\frac{a^2B_1}{1.2^2.3},$$

$$\&c. \qquad \&c.$$

But since we have introduced the arbitrary constant c in log cx, we may assume for B_1 the value zero, and then we have

$$y=-A\left\{\frac{1}{a}-\frac{3a}{1^3.2^2}x^2+\frac{14a^2}{1^3.2^3.3^2}x^3-\&c.\right\}$$

$$+A\left\{x-\frac{ax^2}{1.2}+\frac{a^2x^3}{1.2^2.3}-\frac{a^3x^4}{1.2^2.3^2.4}+\&c.\right\}\log cx.$$

Euler, *Ib.* p. 156.

(3) Let $\dfrac{d^2y}{dx^2}+\dfrac{ay}{x^{\frac{3}{2}}}=0.$

Here $v=A\left\{x-\dfrac{4a}{1.3}x^{\frac{3}{2}}+\dfrac{16a^2}{1.2.3.4}x^2-\dfrac{64a^3}{1.2.3^2.4.5}x^{\frac{5}{2}}+\&c.\right\}$

$$\text{and }\ y=-A\left\{\frac{1}{4a^2}+\frac{x^{\frac{1}{2}}}{a}-\frac{8.4.a}{1^2.3^2}x^{\frac{3}{2}}+\frac{100.16.a^2}{1^2.3^2.2^2.4^2}x^2-\&c.\right\}$$

$$+A\left\{x-\frac{4a}{1.3}x^{\frac{3}{2}}+\frac{16a^2}{1.2.3.4}x^2-\frac{64a^3}{1.2.3^2.4.5}x^{\frac{5}{2}}+\&c.\right\}\log cx.$$

Euler, *Ib.* p. 159.

(4) From the equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0,$$

we easily obtain a particular integral. For if we differentiate the equation r times, we have

$$x\frac{d^{r+2}y}{dx^{r+2}}+(r+1)\frac{d^{r+1}y}{dx^{r+1}}+\frac{d^ry}{dx^r}=0,$$

and when $x=0$

$$\frac{d^{r+1}y}{dx^{r+1}}=-\frac{1}{r+1}\,\frac{d^ry}{dx^r}.$$

Thus any one of the coefficients in Maclaurin's Theorem is derived from the preceding one. Let the first coefficient, or the value of y when $x=0$, be A, then we find as the particular integral

$$y=A\left\{1-\frac{x}{1^2}+\frac{x^2}{1^2.2^2}-\frac{x^3}{1^2.2^2.3^2}+\frac{x^4}{1^2.2^2.3^2.4^2}-\&c.\right\}$$

Let this be put equal to v: then assuming

$$y=u+v\log cx,$$

$$\text{and } u=B+B_1x+B_2x^2+B_3x^3+\&c.,$$

we find by substitution in the given equation

$$u=2A\left(x-\frac{3x^2}{2^3}+\frac{11x^3}{2^3.3^3}-\frac{51x^4}{2^3.3^3.4^3}+\&c.\right)+\frac{B}{A}v.$$

Hence we have

$$y=2A\left(x-\frac{3x^2}{2^3}+\frac{11x^3}{2^3.3^3}-\frac{51x^4}{2^3.3^3.4^3}+\&c.\right)$$

$$+\left(1-\frac{x}{1^2}+\frac{x^2}{1^2.2^2}-\frac{x^3}{1^2.2^2.3^2}+\frac{x^4}{1^2.2^2.3^2.4^2}-\&c.\right)\log cx,$$

the constant $\frac{B}{A}$ being included in c.

Fourier, *Traité de la Chaleur*, p. 372.

(5) Let $$x^r\frac{d^ry}{dx^r}-c^ry=0.$$

The assumption $y = \Sigma(a_n x^n)$ gives

$$\{n(n-1)\ldots(n-r+1)-c^r\}\, a_n = 0.$$

From this it appears that $a_n = 0$ except for those values of n which cause the other factor to vanish. These values of n are r in number; let them be n_1, $n_2 \ldots n_r$, then, the corresponding values of a_n being indeterminate, we have

$$y = C_1 x^{n_1} + C_2 x^{n_2} + \&c. + C_r x^{n_r}.$$

(6) Let $$\frac{d^2y}{dx^2} - \frac{c^2y}{x^4} = 0.$$

Assume $$y = \Sigma(a_n x^n);$$

then $$\frac{d^2y}{dx^2} = \Sigma\{n(n-1)\, a_n x^{n-2}\},$$

$$\frac{c^2y}{x^4} = c^2\Sigma(a_{n+2} x^{n-2}).$$

Substituting these in the equation and equating to zero the coefficients of x^{n-2}, we have

$$n(n-1)\, a_n = c^2 a_{n+2}.$$

If $n = 0$, or $n = 1$, a_2 and a_3 both vanish, and so consequently do all the superior coefficients.

If $n = -1$, $1.2a_{-1} = c^2 a_1$ and $a_{-1} = \dfrac{c^2 a_1}{1.2}$,

$n = -2$, $2.3a_{-2} = c^2 a_0$ and $a_{-2} = \dfrac{c^2 a_0}{1.2.3}$,

$n = -3$, $3.4a_{-3} = c^2 a_{-1}$ and $a_{-3} = \dfrac{c^4 a_1}{1.2.3.4}$,

$n = -4$, $4.5a_{-4} = c^2 a_{-2}$ and $a_{-4} = \dfrac{c^4 a_0}{1.2.3.4.5}$,

&c. &c.

Hence we have

$$y = a_1\left(x + \frac{1}{1.2}\frac{c^2}{x} + \frac{1}{1.2.3.4}\frac{c^4}{x^3} + \&c.\right)$$

$$+ a_0\left(1 + \frac{1}{1.2.3}\frac{c^2}{x^2} + \frac{1}{1.2.3.4.5}\frac{c^4}{x^4} + \&c.\right),$$

a_1 and a_0 being two arbitrary constants.

This may obviously be put under the form

$$y = x\left(A\epsilon^{\frac{c}{x}} + B\epsilon^{-\frac{c}{x}}\right),$$

if $A = \frac{1}{2}\left(a_1 + \frac{a_0}{c}\right)$, and $B = \frac{1}{2}\left(a_1 - \frac{a_0}{c}\right)$.

Euler, *Ib.* p. 166.

(7) Let $$\frac{d^2y}{dx^2} + \frac{c^2y}{x^4} = 0,$$

$$y = x\left(C\cos\frac{c}{x} + C_1\sin\frac{c}{x}\right).$$

Euler, *Ib.* p. 167.

(8) Let $$\frac{d^2y}{dx^2} - \frac{c^2}{x^{\frac{4}{3}}}y = 0.$$

The integral is

$$y = \left(x^{\frac{1}{3}} - \frac{1}{3c}\right)A\epsilon^{3cx^{\frac{1}{3}}} + \left(x^{\frac{1}{3}} + \frac{1}{3c}\right)B\epsilon^{-3cx^{\frac{1}{3}}}.$$

Euler, *Ib.* p. 168.

(9) Let $$\frac{d^2y}{dx^2} - c^2x^{-\frac{8}{5}}y = 0.$$

The integral is

$$y = \left(x^{\frac{2}{5}} - \frac{3}{5c}x^{\frac{1}{5}}\right)A\epsilon^{5cx^{\frac{1}{5}}} + \frac{3}{5c}\left(x^{\frac{1}{5}} + \frac{3}{5c}\right)B\epsilon^{-5cx^{\frac{1}{5}}}.$$

Generally, the integral of

$$\frac{d^2y}{dx^2} - c^2x^{-4\lambda}y = 0$$

will be expressed in finite terms when $\lambda = \frac{r}{2r \pm 1}$.

Mr Leslie Ellis has given (*Cambridge Mathematical Journal*, Vol. II. p. 169 and p. 193) some remarkable methods for reducing to finite functions the solutions in infinite series of certain classes of Differential Equations.

Let the equation be of the form

$$\frac{d^2y}{dx^2} + q^2y = p(p-1)\frac{y}{x^2}. \qquad (1).$$

Then on assuming $y = \Sigma(a_n x^n)$, and substituting in the given equation we obtain as the condition for determining the coefficients

$$\{n(n-1) - p(p-1)\} a_n + q^2 a_{n-2} = 0 \text{......}(2).$$

Now $n(n-1) - p(p-1) = (n-p)(n+p-1) \text{...}(3);$

$$\text{therefore} \quad (n-p)(n+p-1) a_n + q_2 a_{n-2} = 0.$$

Assume $\quad (n+p-1) a_n = (n-p+2) b_n;$

$$\text{then} \quad a_{n-2} = \frac{n-p}{n+p-3} b_{n-2},$$

$$\text{and} \quad (n-p+2)(n+p-3) b_n + q^2 b_{n-2} = 0 \text{...}(4).$$

Again assume $(n+p-3) b_n = (n-p+4) c_n$, and so on in succession. We shall thus obtain a series of equations of which the type is

$$(n-p+\mu)(n+p-\mu-1) l_n + q^2 l_{n-2} = 0 \text{......}(5),$$

μ being an even number.

If p be even let $p = \mu$, then $p - \mu - 1 = -1$.

If p be odd let $p = \mu + 1$, then $\mu - p = -1$.

In both cases the equation (5) becomes

$$n(n-1) l_n + q^2 l_{n-2} = 0.$$

This is the relation between the coefficients which we should obtain from the equation

$$\frac{d^2 y}{dx^2} + q^2 y = 0. \text{.........................} (6).$$

Hence $\quad \Sigma(l_n x^n) = C \sin(qx + a) \text{ ....} (7),$

that being the integral of equation (6).

Now suppose

$$(n-p+\mu-2)(n+p-\mu+1) i_n + q^2 i_{n-2} = 0,$$

$$(n-p+\mu)(n+p-\mu-1) k_n + q^2 k_{n-2} = 0,$$

to be any two consecutive equations; then

$$(n+p-\mu+1) i_n = (n-p+\mu) k_n \text{ ....} (8),$$

but $\quad n - p + \mu = n + p - \mu + 1 - 2(p-\mu) - 1;$

therefore $$i_n = k_n - \frac{\{2(p-\mu)+1\}k_n}{n+p-\mu+1},$$

and $$\frac{k_n}{n+p-\mu+1} = -\frac{1}{q^2}(n+2-p+\mu)k_{n+2};$$

therefore

$$\Sigma(i_n x^n) = \Sigma(k_n x^n) + \frac{2(p-\mu)+1}{q^2}\Sigma\{(n-p+\mu)k_n x^{n-2}\}.$$

Now $(n-p+\mu)k_n x^{n-2} = x^{p-\mu-1}(n-p+\mu)k_n x^{n-p+\mu-1}$

$$= x^{p-\mu-1}\frac{d}{dx}\left(\frac{k_n x^n}{x^{p-\mu}}\right);$$

therefore

$$\Sigma(i_n x^n) = \Sigma(k_n x^n) + \frac{2(p-\mu)+1}{q^2}x^{p-\mu-1}\frac{d}{dx}\Sigma\left(\frac{k_n x^n}{x^{p-\mu}}\right).$$

By the application of this formula y or $\Sigma(a_n x_n)$ may be deduced by a series of regular operations from $C\sin(qx+a)$.

If p be even $2(p-\mu)+1$ gives the series 1, 5, 9.........

If p be odd it gives the series 3, 7, 11.........

(10) Let $$\frac{d^2y}{dx^2} + q^2y = \frac{2y}{x^2},$$

where $p = 2$. The integral is

$$y = C\left\{\sin(qx+a) + \frac{1}{qx}\cos(qx+a)\right\}.$$

(11) Let $$\frac{d^2y}{dx^2} + q^2y = \frac{6y}{x^2},$$

where $p = 3$. The integral is

$$y = C\left\{\sin(qx+a)\left(1-\frac{3}{q^2x^2}\right) + \frac{3}{qx}\cos(qx+a)\right\}.$$

This method may be successfully applied to reduce

$$\frac{d^my}{dx^m} + q^my = p(p-1)\frac{1}{x^2}\frac{d^{m-2}y}{dx^{m-2}},$$

when p or $p-1$ is divisible by m.

(12) Let $\dfrac{d^3y}{dx^3} + q^3y = \dfrac{6}{x^2}\dfrac{dy}{dx}$.

The complete integral is

$$y = C_1 \epsilon^{-qx}\left(1 + \frac{2}{qx}\right)$$

$$+ C_2 \epsilon^{\frac{qx}{2}}\left\{\sin\left(\frac{3^{\frac{1}{2}}}{2}qx + \alpha\right)\left(1 - \frac{1}{qx}\right) + \frac{3^{\frac{1}{2}}}{qx}\cos\left(\frac{3^{\frac{1}{2}}}{2}qx + \alpha\right)\right\}.$$

(13) Let $\dfrac{d^2y}{dx^2} + q\dfrac{dy}{dx} = \dfrac{2y}{x^2}$.

This equation presents a peculiarity, inasmuch as if we neglect a factor, which apparently disappears, we shall have a solution which is erroneous or incomplete.

Assume $y = \Sigma(a_n x^n)$, then

$\{n(n-1) - 2\}a_n + (n-1)qa_{n-1} = 0$,

or $(n-2)(n+1)a_n + (n-1)qa_{n-1} = 0$ .... (1).

Let $(n+1)a_n = (n-1)b_n$ .... (2),

then $(n-2)(n-1)nb_n + (n-2)(n-1)qb_{n-1} = 0$ .... (3).

The factor $(n-2)$ may be safely neglected, but $(n-1)$ must be retained, as it enters into the solution of the auxiliary equation

$$\frac{d^2z}{dx^2} + q\frac{dz}{dx} = 0.$$

From (2) we have

$$a_n = b_n - \frac{2b_n}{n+1};$$

and as, except when $n = 1$, we have $nb_n + qb_{n-1} = 0$,

$$\frac{b_n}{n+1} = -\frac{1}{q}b_{n+1}, \quad \text{except when } n = 0;$$

therefore $a_n = b_n + \dfrac{2}{q}b_{n+1}$ .... (4).

The solution of the auxiliary equation is

$$z = C_1 + C_2\epsilon^{-qx},$$

and from (4) it appears that

$$y=\left(1+\frac{2}{qx}\right)z=\left(1+\frac{2}{qx}\right)(C_1+C_2\epsilon^{-qx}).$$

This appears to be the solution of the equation, but it does not satisfy it unless $C_1=0$, when it becomes

$$y=C_2\left(1+\frac{2}{qx}\right)\epsilon^{-qx},$$

which is only a particular integral, and therefore incomplete.

This arises from our implying in the use of equation (4) that $nb_n+qb_{n-1}=0$ is generally true, whereas the equation

$$(n-1)(nb_n+qb_{n-1})=0,$$

derived from the auxiliary equation

$$\frac{d^2z}{dx^2}+q\frac{dz}{dx}=0,$$

shews that b_1 is not necessarily connected with b_0, since it may be satisfied by $n=1$.

To complete the solution, we have from (2) which is always true

$$a_0=-b_0,$$

and from (4) which is true for $n=-1$, we have

$$a_{-1}=b_{-1}+\frac{2}{q}b_0,$$

$$\text{or as}\quad b_{-1}=0,\quad a_{-1}=\frac{2}{q}b_0=-\frac{2}{q}a_0.$$

These quantities are independent of a_1, a_2, &c., therefore writing C_1 for a_0 as it is an arbitrary constant,

$$y=C_1\left(1-\frac{2}{qx}\right)$$

is a particular integral of the proposed equation, and the complete solution is

$$y=C_1\left(1-\frac{2}{qx}\right)+C_2\left(1+\frac{2}{qx}\right)\epsilon^{-qx}.$$

(14) To integrate

$$\frac{d^m y}{dx^m} + k^m y = \frac{pm}{x}\frac{d^{m-1}y}{dx^{m-1}},$$

where p is an integer.

Assume $y = \Sigma(a_n x^n)$, then if

$$\Sigma(a_n x^n) = x^{m(p+1)-1}\left(\frac{1}{x^{m-1}}\frac{d}{dx}\right)^p \frac{1}{x^{m-1}}\Sigma(b_n x^n),$$

$$a_n = (n - pm + 1) \dots (n - m + 1)\, b_n.$$

But from the given equation

$$n(n-1) \dots (n-m+2)\,\{n - m(p+1) + 1\}\, a_n + k^m a_{n-m} = 0,$$

from which

$$n(n-1) \dots (n-m+2) \dots (n-m+1)\, b_n + k^m b_{n-m} = 0.$$

But this is the equation which would result from substituting $\Sigma(b_n x^n)$ in

$$\frac{d^m y}{dx^m} + k^m y = 0;$$

therefore $\Sigma(b_n x^n)$ is the solution of this last equation, and is therefore known. Calling it X, we have

$$y = \Sigma(a_n x^n) = x^{m(p+1)-1}\left(\frac{1}{x^{m-1}}\frac{d}{dx}\right)^p \frac{X}{x^{m-1}}.$$

Let $m = 2$, $p = 2$, then the integral of

$$\frac{d^2 y}{dx^2} - \frac{4}{x}\frac{dy}{dx} + k^2 y = 0$$

is $$y = x^5\left(\frac{1}{x}\frac{d}{dx}\right)^2 C\cos\frac{(kx+a)}{x};$$

or $$y = C\left\{(3 - k^2x^2)\cos(kx+a) + 3kx\sin(kx+a)\right\}.$$

Ellis, *Cam. Math. Jour.* Vol. II. p. 202.

CHAPTER VI.

PARTIAL DIFFERENTIAL EQUATIONS.

SECT. 1. *Linear equations with constant coefficients.*

BY the method of the separation of symbols the integration of Linear Partial Differential Equations is reduced to the same processes as those for the integration of ordinary differential equations of the same class. Hence the theory which is given in the beginning of Chap. IV. is equally applicable to the present subject, and it is unnecessary to repeat it here; I shall therefore content myself with referring to what has been previously said in the Chapter alluded to, adding that every differential equation of this class between two variables has an exact analogue among partial differential equations of the same class, and that the form of the solution of the latter is the same as that of the former. On this point one remark may be made which is of considerable importance in the interpretation of our results. As in the solution of ordinary differential equations we continually meet with expressions of the form

$$C\epsilon^{ax},$$

so in partial differential equations we shall find expressions of the form

$$\epsilon^{a\frac{d}{dy}\cdot x}\phi(y),$$

in which the arbitrary function takes the place of the arbitrary constant. Now as the preceding formula is the symbolical expression for Taylor's Theorem, we know that

$$\epsilon^{a\frac{d}{dy}\cdot x}\phi(y) = \phi(y + ax).$$

Hence, in the solution of partial differential equations, arbitrary functions of binomials play the same parts as arbitrary constants multiplied by exponentials do in equations between two variables.

(1) Let the equation be

$$a\frac{dz}{dx} + b\frac{dz}{dy} = c.$$

This may be put under the form

$$\left(a\frac{d}{dx} + b\frac{d}{dy}\right)z = c;$$

whence $$z = \left(a\frac{d}{dx} + b\frac{d}{dy}\right)^{-1} c.$$

Now supposing x to be the independent variable, and $\frac{d}{dy}$ a constant, with respect to it, by the Theorem given in Ex. (11), Chap. xv. of the Diff. Calc. this is equivalent to

$$z = \frac{1}{a}\epsilon^{-\frac{b}{a}x\frac{d}{dy}} \int dx\, \epsilon^{\frac{b}{a}x\frac{d}{dy}} \cdot c;$$

or, effecting the integration, and adding an arbitrary function of y, instead of an arbitrary constant,

$$z = c\frac{x}{a} + \frac{1}{a}\epsilon^{-\frac{b}{a}x\frac{d}{dy}}\phi(y).$$

Now by Taylor's Theorem

$$\epsilon^{-\frac{b}{a}x\frac{d}{dy}}\phi(y) = \phi\left(y - \frac{b}{a}x\right);$$

or, as the form of ϕ is arbitrary, we may for

$\frac{1}{a}\phi\left(y - \frac{b}{a}x\right)$ write $\phi(ay - bx)$, so that

$$z = \frac{cx}{a} + \phi(ay - bx).$$

It is obvious that if we had taken y for our independent variable, and considered $\frac{d}{dx}$ as a constant with respect to it, we should have had

$$z = \frac{cy}{b} + \phi(bx - ay).$$

(2) Let $\frac{dz}{dx} - a\frac{dz}{dy} = \epsilon^{mx}\cos ry$,

$$z = \left(\frac{d}{dx} - a\frac{d}{dy}\right)^{-1}\epsilon^{mx}\cos ry$$

$$= \epsilon^{ax\frac{d}{dy}}\int dx\,\epsilon^{-ax\frac{d}{dy}}\epsilon^{mx}\cos ry.$$

But by Taylor's Theorem

$$\epsilon^{-ax\frac{d}{dy}}\cos ry = \cos r(y - ax)\text{; therefore}$$

$$z = \epsilon^{ax\frac{d}{dy}}\int dx\,\epsilon^{mx}\cos r(y - ax);$$

and, integrating with respect to x,

$$z = \epsilon^{ax\frac{d}{dy}}\epsilon^{mx}\frac{\{m\cos r(y - ax) - ar\sin r(y - ax)\}}{m^2 + a^2r^2} + \epsilon^{ax\frac{d}{dy}}\phi(y);$$

or, $$z = \epsilon^{mx}\frac{(m\cos ry - ar\sin ry)}{m^2 + a^2r^2} + \phi(y + ax).$$

The same method is applicable to any number of independent variables.

(3) Let the equation be

$$\frac{du}{dx} + b\frac{du}{dy} + c\frac{du}{dz} = xyz;$$

or, $$\left(\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right)u = xyz;$$

whence

$$u = \left(\frac{d}{dx} + b\frac{d}{dy} + c\frac{d}{dz}\right)^{-1}xyz + \text{complementary function.}$$

If we expand the operating factor in ascending powers of $\left(b\frac{d}{dy} + c\frac{d}{dz}\right)$ we shall have

$$u = \left(\frac{d}{dx}\right)^{-1}\left\{1 - \left(\frac{d}{dx}\right)^{-1}\left(b\frac{d}{dy} + c\frac{d}{dz}\right) + 2bc\left(\frac{d}{dx}\right)^{-2}\frac{d}{dy}\frac{d}{dz}\right\}xyz;$$

the other terms being neglected, because when the operations are performed they vanish of themselves. The complementary function in this case is

$$\epsilon^{-x\left(b\frac{d}{dy}+c\frac{d}{dz}\right)}\phi(y, z) = \phi(y-bx, z-cx);$$

therefore, effecting the operations indicated,

$$u = \frac{x^2}{2}yz - \frac{x^3}{6}(bz+cy) + bc\frac{x^4}{12} + \phi(y-bx, z-cx).$$

(4) Let $\frac{dz}{dt} = a\frac{d^2z}{dx^2}$; or $\left\{\frac{d}{dt} - a\left(\frac{d}{dx}\right)^2\right\}z = 0$;

$$\text{therefore } z = \left(\frac{d}{dt} - a\frac{d^2}{dx^2}\right)^{-1} 0.$$

If we integrate with respect to t we find

$$z = \epsilon^{at\frac{d^2}{dx^2}}\phi(x) = \phi(x) + at\frac{d^2\phi(x)}{dx^2} + \frac{a^2t^2}{1.2}\frac{d^4\phi(x)}{dx^4} + \&c.$$

If we integrate with respect to x we shall have two arbitrary functions of t, since the differential with respect to the former variable is of the second order.

Writing the equation in the form

$$\frac{d^2z}{dx^2} - \frac{1}{a}\frac{dz}{dt} = 0,$$

we may divide it into two factors

$$\left\{\frac{d}{dx} - \frac{1}{a^{\frac{1}{2}}}\left(\frac{d}{dt}\right)^{\frac{1}{2}}\right\}\left\{\frac{d}{dx} + \frac{1}{a^{\frac{1}{2}}}\left(\frac{d}{dt}\right)^{\frac{1}{2}}\right\}z = 0.$$

$$\text{Whence } z = \epsilon^{\left(\frac{1}{a}\frac{d}{dt}\right)^{\frac{1}{2}}x}\phi(t) + \epsilon^{-\left(\frac{1}{a}\frac{d}{dt}\right)^{\frac{1}{2}}x}\psi(t);$$

or, if we put

$$\phi(t) + \psi(t) = F(t), \text{ and } \left(\frac{d}{dt}\right)^{\frac{1}{2}}\{\phi(t) - \psi(t)\} = f(t),$$

this may be put under the form

$$z = F(t) + \frac{1}{a}\frac{x^2}{1.2}\frac{dF(t)}{dt} + \frac{1}{a^2}\frac{x^4}{1.2.3.4}\frac{d^2F(t)}{dt^2} + \&c.$$

$$+ xf(t) + \frac{1}{a}\frac{x^3}{1.2.3}\frac{df(t)}{dt} + \frac{1}{a^2}\frac{x^5}{1.2.3.4.5}\frac{d^2f(t)}{dt^2} + \&c.$$

It seems anomalous that the same equation should admit of two solutions differing so essentially in character that the one contains two arbitrary functions and the other only one: but the following considerations may serve to explain the difficulty. Since by Maclaurin's theorem any function of a variable may be expressed by means of its differential coefficients, taken with respect to that variable, we know the function if we can determine its successive differential coefficients. Now from the equation

$$\frac{dz}{dt} = a\frac{d^2z}{dx^2},$$

we can determine the values of all the differential coefficients with respect to t, when $t = 0$, if we know the value of z when $t = 0$. This therefore is the only undetermined quantity in this case, and it corresponds to the arbitrary function $\phi(x)$. But from the equation

$$\frac{d^2z}{dx^2} = \frac{1}{a}\frac{dz}{dt},$$

we can only, from the value of z when $x = 0$, determine the values of the alternate differential coefficients: and in order to determine the others we must also know the value of $\dfrac{dz}{dx}$ when $x = 0$. Therefore in this case there are two indeterminate quantities corresponding to the arbitrary functions $F(t)$ and $f(t)$.

This is the equation for determining the linear transmission of heat in an infinite solid.

Fourier, *Traité de la Chaleur*, p. 471 and p. 509.

(5) Let $$\frac{dz}{dt} = a\frac{d^2z}{dx^2} - bz;$$

or $$\left(\frac{d}{dt} + b - a\frac{d^2}{dx^2}\right) z = 0.$$

Whence $$z = \epsilon^{-bt}\epsilon^{a\frac{d^2}{dx^2}} f(x).$$

This is the equation for determining the motion of heat in a ring.

Fourier, *Ib.* p. 266.

(6) Let $$\frac{d^2z}{dt^2} - a^2\frac{d^2z}{dx^2} = 0.$$

The operating factor in this case may be decomposed into two, and the equation then becomes

$$\left(\frac{d}{dt} - a\frac{d}{dx}\right)\left(\frac{d}{dt} + a\frac{d}{dx}\right) z = 0.$$

Whence $z = \epsilon^{at\frac{d}{dx}}\phi(x) + \epsilon^{-at\frac{d}{dx}}\psi(x)$;

or $z = \phi(x + at) + \psi(x - at)$.

This is one of the most important equations in the application of mathematics to Natural Philosophy, being that which results from the investigation of the motion of vibrating chords, and of the pulses produced by a disturbance in a small cylindrical column of air.

(7) Let $$\frac{d^2z}{dx^2} - a^2\frac{d^2z}{dy^2} = xy.$$

The complementary function in this example is the same as the integral of the last, and the result of the inverse operation on xy will best be found by expanding in ascending powers of $\left(\frac{d}{dy}\right)^2$, when it is easy to see that all the terms after the first may be neglected. We find accordingly

$$z = \frac{x^3y}{6} + \phi(y + ax) + \psi(y - ax).$$

(8) Let $$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = \cos mx \cos ny.$$

The solution of this is

$$z = -\frac{\cos mx \cos ny}{m^2 + n^2} + \cos\left(x\frac{d}{dy}\right)\phi(y) + \sin\left(x\frac{d}{dy}\right)\psi(y).$$

Compare this with Chap. IV. Sect. 1, Ex. (8).

(9) The equation

$$\frac{d^2z}{dx^2} - 2a\frac{d^2z}{dx\,dy} + a^2\frac{d^2z}{dy^2} = 0,$$

may be put under the form

$$\left(\frac{d}{dx} - a\frac{d}{dy}\right)^2 z = 0,$$

so that the two factors are equal. Hence, integrating with respect to x,

$$z = \epsilon^{ax\frac{d}{dy}} \int^2 dx^2 . 0 = \epsilon^{ax\frac{d}{dy}} \{x\phi(y) + \psi(y)\};$$

$$\text{or} \quad z = x\phi(y + ax) + \psi(y + ax).$$

(10) Let $\dfrac{d^2z}{dx^2} - a^2\dfrac{d^2z}{dy^2} + 2ab\dfrac{dz}{dx} + 2a^2b\dfrac{dz}{dy} = 0.$

When resolved into its factors this becomes

$$\left\{\frac{d}{dx} - \left(a\frac{d}{dy} - 2ab\right)\right\}\left(\frac{d}{dx} + a\frac{d}{dy}\right) z = 0.$$

Integrating with respect to the first factor we have

$$\left(\frac{d}{dx} + a\frac{d}{dy}\right) z = \epsilon^{x\left(a\frac{d}{dy} - 2ab\right)} \phi(y).$$

The effect of the second factor on the second side of the equation will be simply to alter the function of y, and as that is arbitrary we may leave it as it stands, so that we have, on adding the complementary function due to the second factor,

$$z = \epsilon^{x\left(a\frac{d}{dy} - 2ab\right)} \phi(y) + \epsilon^{-ax\frac{d}{dy}} \psi(y),$$

$$\text{or } z = \epsilon^{-2abx}\phi(y + ax) + \psi(y - ax).$$

Euler, *Calc. Integ.* Vol. III. p. 210.

(11) Let $\dfrac{d^2z}{dxdy} + a\dfrac{dz}{dx} + b\dfrac{dz}{dy} + abz = V.$

$$\text{or} \quad \left(\frac{d}{dy} + a\right)\left(\frac{d}{dx} + b\right) z = V.$$

Whence $z = \epsilon^{-(ay+bx)} \int dy\, \epsilon^{ay} \int dx\, \epsilon^{bx} V + \epsilon^{-ay}\phi(x) + \epsilon^{-bx}\psi(y).$

If $V = \epsilon^{my+nx}$, then

$$z = \frac{e^{my+nx}}{(m+a)(n+b)} + e^{-ay}\phi(x) + e^{-bx}\psi(y).$$

Euler, *Ib.* p. 189.

(12) Let $\dfrac{d^2z}{dxdy} - az = 0$.

This gives $z = \left(\dfrac{d}{dx}\cdot\dfrac{d}{dy} - a\right)^{-1} 0$,

or $z = \left(\dfrac{d}{dy}\right)^{-1} e^{ax\left(\frac{d}{dy}\right)^{-1}} \phi(y) = e^{ax\left(\frac{d}{dy}\right)^{-1}} \int \phi(y)\,dy.$

(13) Let $\dfrac{d^3z}{dx^3} - a^3z = 0$.

The roots of the equation

$$u^3 - a^3 = 0$$

are a, $a\left\{\cos\dfrac{2\pi}{3} + (-)^{\frac{1}{2}}\sin\dfrac{2\pi}{3}\right\}$, and $a\left\{\cos\dfrac{2\pi}{3} - (-)^{\frac{1}{2}}\sin\dfrac{2\pi}{3}\right\}$.

Therefore

$$z = e^{ax}\phi(y) + e^{-\frac{ax}{2}}\cos\left(\frac{a3^{\frac{1}{2}}x}{2}\right)\psi_1(y) + e^{-\frac{ax}{2}}\sin\left(\frac{a3^{\frac{1}{2}}x}{2}\right)\psi_2(y).$$

(14) Let

$$\frac{d^3z}{dx^3} - (2a+b)\frac{d^3z}{dx^2dy} + (2ab+a^2)\frac{d^3z}{dxdy^2} - a^2b\frac{d^3z}{dy^3} = 0.$$

In this case two of the operating factors are equal, and we find

$$z = \psi(y+bx) + f(y+ax) + xf_1(y+ax).$$

(15) Let the equation contain three independent variables, as in

$$\frac{d^3u}{dx^2dy} - 2\frac{d^3u}{dxdy^2} - 3\frac{d^3u}{dx^2dz} - 3\frac{d^3u}{dxdz^2}$$

$$- 2\frac{d^3u}{dy^2dz} + 6\frac{d^3u}{dydz^2} + 7\frac{d^3u}{dxdydz} = 0.$$

When decomposed into its factors this becomes

$$\left(\frac{d}{dx} - 2\frac{d}{dy}\right)\left(\frac{d}{dx} + \frac{d}{dz}\right)\left(\frac{d}{dy} - 3\frac{d}{dz}\right)u = 0;$$

the integral of which is

$$u = f(y + 2x, z) + \phi(y, z - x) + \psi(x, z + 3y).$$

(16) Take the general equation with two independent variables

$$\frac{d^n z}{dx^n} + A_1\frac{d^n z}{dx^{n-1}dy} + A_2\frac{d^n z}{dx^{n-2}dy^2} + \text{\&c.} + A_n\frac{d^n z}{dy^n} = V,$$

where the index of differentiation is the same in every term, and the coefficients are constants, and V is a function of x and y. When decomposed into factors it takes the form

$$\left(\frac{d}{dx} - a_1\frac{d}{dy}\right)\left(\frac{d}{dx} - a_2\frac{d}{dy}\right) \dots\dots \left(\frac{d}{dx} - a_n\frac{d}{dy}\right) z = V,$$

where a_1, $a_2 \dots\dots a_n$ are the roots of the equation

$$u^n + A_1 u^{n-1} + A_2 u^{n-2} + \text{\&c.} + A_n = 0.$$

Now in decomposing the inverse operation into partial fractions each of the coefficients involves $\left(\frac{d}{dy}\right)^{-(n-1)}$ as a factor in the numerator, since the denominators consist of the products of $(n - 1)$ factors of the form

$$(a_1 - a_2)\frac{d}{dy}.$$

Hence giving to N_1, N_2, &c. the same meanings as in Ex. (6) of Chap. xv. of the Diff. Calc.

$$z = N_1\left(\frac{d}{dx} - a_1\frac{d}{dy}\right)^{-1}\left(\frac{d}{dy}\right)^{-(n-1)}V + N_2\left(\frac{d}{dx} - a_2\frac{d}{dy}\right)^{-1}\left(\frac{d}{dy}\right)^{-(n-1)}V$$

$$+ \text{\&c.} + N_n\left(\frac{d}{dx} - a_n\frac{d}{dy}\right)^{-1}\left(\frac{d}{dy}\right)^{-(n-1)}V.$$

If for shortness we represent $\left(\frac{d}{dy}\right)^{-(n-1)}V$ by V_1, and if we transform the operating factors by the formula

$$\left(\frac{d}{dx} - a\frac{d}{dy}\right)^{-1} = \epsilon^{ax\frac{d}{dy}}\int dx\, \epsilon^{-ax\frac{d}{dy}},$$

we have

$$z = N_1 e^{a_1 x \frac{d}{dy}} \int dx\, e^{-a_1 x \frac{d}{dy}} V_1 + N_2 e^{a_2 x \frac{d}{dy}} \int dx\, e^{-a_2 x \frac{d}{dy}} V_1$$

$$+ \&c. + \&c. + N_n e^{a_n x \frac{d}{dy}} \int dx\, e^{-a_n x \frac{d}{dy}} V_1.$$

The complementary functions are supposed to be included under the signs of integration: if we wish to see their form we have merely to suppose $V_1 = 0$ in the above expressions, when after obvious transformations we find

$$z = f_1(y + a_1 x) + f_2(y + a_2 x) + \&c. + f_n(y + a_n x).$$

(17) Let $\dfrac{d^2 z}{dx^2} + 3\dfrac{d^2 z}{dx\,dy} + 2\dfrac{d^2 z}{dy^2} = x + y.$

In this case $a_1 = -1$, $a_2 = -2$, $N_1 = 1$, $N_2 = -1$.

Also $$\left(\frac{d}{dy}\right)^{-1}(x + y) = \frac{(x + y)^2}{2};$$

therefore $$z = \frac{(x + y)^3}{36} + f_1(y - x) + f_2(y - 2x).$$

SECT. 2. *Equations in which the coefficients are functions of the independent variables.*

As in the case of the similar class of ordinary differential equations these equations may sometimes be reduced to forms in which the coefficients are constant. Thus, equations of the first degree of the form

$$\frac{dz}{dx} + XY\frac{dz}{dy} = Pz + Q,$$

where X is a function of x only, Y a function of y only, and P and Q functions of both x and y, may be reduced to the form

$$\frac{dz}{dx} + X\frac{dz}{dy'} = Pz + Q,$$

by assuming $dy' = \dfrac{dy}{Y}$. This equation may be written

$$\frac{dz}{dx} + \left(X\frac{d}{dy'} - P\right)z = Q,$$

in which shape it is seen to be a differential equation of the first order with respect to x with coefficients which are functions of that variable; it may therefore be integrated by the method of Chap. IV. Sect. 2.

We may sometimes however reduce the equation at once to constant coefficients by changing both the independent variables at once.

Greatheed, *Philosophical Magazine*, Sept. 1837.

Ex. (1) Let $x\dfrac{dz}{dx}+y\dfrac{dz}{dy}=nz$.

By assuming $\dfrac{dx}{x}=du$, $\dfrac{dy}{y}=dv$, this becomes

$$\left(\frac{d}{du}+\frac{d}{dv}-n\right)z=0.$$

Integrating with respect to u, we have

$$z=\epsilon^{-u\left(\frac{d}{dv}-n\right)}\phi(v)=\epsilon^{nu}\epsilon^{-u\frac{d}{dv}}\phi(v);$$

or
$$z=\epsilon^{nu}\phi(v-u).$$

But $u=\log x$, $v=\log y$; therefore

$$v-u=\log\left(\frac{y}{x}\right), \text{ and } \phi(v-u)=\phi\log\left(\frac{y}{x}\right)=f\left(\frac{y}{x}\right),$$

so that
$$z=x^nf\left(\frac{y}{x}\right).$$

If we had integrated with respect to v, we should have found

$$z=y^nf\left(\frac{x}{y}\right).$$

The interpretation of these results is that z is a homogeneous function of n dimensions in x and y. This is obvious, as the differential equation is the condition of homogeneity of a function of two variables.

(2) Let $x\dfrac{dz}{dx}-y\dfrac{dz}{dy}=\dfrac{x^2}{y}$.

Changing the variables, as in the last example, we have

$$\left(\frac{d}{du}-\frac{d}{dv}\right)z=\epsilon^{2u}\epsilon^{-v}.$$

The integral of this is

$$z = \frac{\epsilon^{3u}\epsilon^{-v}}{3} + \phi(u+v);$$

or

$$z = \frac{x^3}{3y} + \psi(xy).$$

(3) Let $\dfrac{dz}{dx} - \dfrac{dz}{dy} = \dfrac{z}{x+y}$.

Integrating with respect to x, we have

$$z = \epsilon^{\int dx\left(\frac{d}{dy}+\frac{1}{x+y}\right)}\phi(y) = \epsilon^{\int dx\frac{d}{dy}}\left\{\epsilon^{-\int dx\frac{d}{dy}}\left(\epsilon^{\int\frac{dx}{x+y}}\right)\phi(y)\right\}.$$

But $\epsilon^{\int dx\frac{d}{dy}} = \epsilon^{x\frac{d}{dy}}$, and $\epsilon^{-x\frac{d}{dy}}\epsilon^{\int\frac{dx}{x+y}} = \epsilon^{\int\frac{dx}{y}} = \epsilon^{\frac{x}{y}}$;

therefore $$z = \epsilon^{x\frac{d}{dy}}\left\{\epsilon^{\frac{x}{y}}\phi(y)\right\} = \epsilon^{\frac{x}{y+x}}\phi(y+x).$$

It is to be observed that, in this method of integration, when we have a function of the form

$$\epsilon^{\int dx\left(\frac{d}{dy}+P\right)}\phi(y),$$

where P is a function of x and y, we must not allow the first term of the exponent of ϵ to act on the second, by putting it under the form

$$\epsilon^{\int dx\frac{d}{dy}}\left\{\epsilon^{\int dx P}\phi(y)\right\}.$$

It is therefore necessary to prefix to the factor $\epsilon^{\int dx P}$ the inverse operation of

$$\epsilon^{\int dx\frac{d}{dy}}, \text{ or } \epsilon^{-\int dx\frac{d}{dy}},$$

so that the expression takes the form

$$\epsilon^{\int dx\frac{d}{dy}}\left\{\epsilon^{-\int dx\frac{d}{dy}}\left(\epsilon^{\int dx P}\right)\phi(y)\right\},$$

where the newly introduced factor acts only on that which immediately precedes it.

(4) Let $\dfrac{dz}{dx} + m\dfrac{dz}{dy} = n\dfrac{z}{y}$.

Integrating with respect to x, we have

$$z = \epsilon^{-\int dx\left(m\frac{d}{dy} - \frac{n}{y}\right)} \phi(y) = \epsilon^{-mx\frac{d}{dy}} \left\{\epsilon^{n\int\frac{dx}{y+mx}} \phi(y)\right\};$$

therefore $$z = y^{\frac{n}{m}} \phi(y - mx).$$

(5) Let $$\frac{1}{x}\frac{dz}{dx} + \frac{1}{y}\frac{dz}{dy} = \frac{z}{y^2}.$$

Put $x\,dx = du$, $y\,dy = dv$; then the equation becomes

$$\frac{dz}{du} + \frac{dz}{dv} = \frac{z}{2v};$$

This is a particular case of the last example, and its integral is

$$z = v^{\frac{1}{2}} \phi(v - u);$$

and therefore $$z = y\,\phi(y^2 - x^2).$$

(6) Let $$y\frac{dz}{dx} + x\frac{dz}{dy} = z.$$

Dividing both sides by xy and putting $u = x^2$, $v = y^2$, we have

$$\frac{dz}{du} + \frac{dz}{dv} = \frac{z}{2(uv)^{\frac{1}{2}}}.$$

Therefore, integrating with respect to u,

$$z = \epsilon^{-\int du\left\{\frac{d}{dv} - \frac{1}{2(uv)^{\frac{1}{2}}}\right\}} \phi(v) = \epsilon^{-u\frac{d}{dv}} \left\{\epsilon^{\int\frac{du}{2(uv+u^2)^{\frac{1}{2}}}} \phi(v)\right\},$$

since $\epsilon^{u\frac{d}{dv}}(uv)^{\frac{1}{2}} = (uv + u^2)^{\frac{1}{2}}$. Hence

$$z = \left\{\tfrac{1}{2}(u + v) + (uv)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \phi(v - u);$$

or, putting for u and v their values, and omitting $2^{-\frac{1}{2}}$ as it may be included in the arbitrary function, we have

$$z = (x + y)\,\phi(y^2 - x^2).$$

(7) Let $$\sec x\frac{dz}{dx} + a\frac{dz}{dy} = z\cot y;$$

The integral is

$$z = (\sin y)^{\frac{1}{a}} \phi(y - a\sin x).$$

(8) Let $x^2 \frac{dz}{dx} + y^2 \frac{dz}{dy} = \frac{x^3}{y}$.

By assuming $\frac{dx}{x^2} = du$, $\frac{dy}{y^2} = dv$, this becomes

$$\frac{dz}{du} + \frac{dz}{dv} = -\frac{v}{u^3}.$$

Integrating with respect to u,

$$z = \frac{v}{2u^2} + \frac{1}{2u} + \epsilon^{-u\frac{d}{dv}} \phi(v).$$

Therefore $z = -\frac{x^2}{2y} - \frac{x}{2} + \phi\left(\frac{y-x}{xy}\right)$.

(9) Let $x \frac{dz}{dx} + (1 + y^2)^{\frac{1}{2}} \frac{dz}{dy} = xy$.

Assuming $\frac{dx}{x} = u$, $\frac{dy}{(1 + y^2)^{\frac{1}{2}}} = dv$ we have

$$\frac{dz}{du} + \frac{dz}{dv} = \frac{\epsilon^u}{2}(\epsilon^v - \epsilon^{-v});$$

whence $$z = \frac{\epsilon^{u+v}}{4} - \frac{u\epsilon^{u-v}}{2} + \phi(v - u).$$

Or, substituting for u and v their values in x and y,

$$z = \frac{x}{4}\{(1 + y^2)^{\frac{1}{2}} + y\} - \frac{x}{2}\{(1 + y^2)^{\frac{1}{2}} - y\} \log x + f\left\{\frac{(1 + y^2)^{\frac{1}{2}} + y}{x}\right\}.$$

(10) Let $x \frac{du}{dx} + y \frac{du}{dy} + z \frac{du}{dz} = au + \frac{xy}{z}$.

Then $u = \frac{xy}{(1-a)z} + z^a \phi\left(\frac{x}{z}, \frac{y}{z}\right)$.

Equations of the second and higher orders may sometimes be reduced by transformations similar to those employed in Chap. IV. Sect. 2.

(11) Let $x^2 \frac{d^2z}{dx^2} + 2xy \frac{d^2z}{dxdy} + y^2 \frac{d^2z}{dy^2} = x^m y^n$.

Putting $\frac{dx}{x} = du$, $\frac{dy}{y} = dv$, this becomes

$$\left\{\left(\frac{d}{du} + \frac{d}{dv}\right)^2 - \left(\frac{d}{du} + \frac{d}{dv}\right)\right\} z = \epsilon^{mu} . \epsilon^{nv}.$$

The integral of this is

$$z = \frac{\epsilon^{mu} . \epsilon^{nv}}{(m+n)(m+n-1)} + \epsilon^u \phi(v-u) + \psi(v-u);$$

$$\text{or } z = \frac{x^m y^n}{(m+n)(m+n-1)} + xF\left(\frac{y}{x}\right) + f\left(\frac{y}{x}\right).$$

(12) Let $x^2 \frac{d^2z}{dx^2} - y^2 \frac{d^2z}{dy^2} = xy.$

By means of the same transformation as in the last example we find

$$z = xy \log x + xF\left(\frac{y}{x}\right) + f(xy).$$

(13) Let $(x+y)\frac{d^2z}{dx\,dy} - a\frac{dz}{dx} = 0.$

If we put $\frac{dz}{dx} = v$ this becomes

$$(x+y)\frac{dv}{dy} - av = 0,$$

the integral of which is $v = (x+y)^a \phi(x)$,

so that $$z = \int dx\,(x+y)^a \phi(x) + \psi(y).$$

(14) Integrate the equation

$$x^n \frac{d^nz}{dx^n} + nx^{n-1}y\frac{d^nz}{dx^{n-1}dy} + \frac{n(n-1)}{1 . 2}\frac{d^nz}{dx^{n-2}dy^2} + \&c. + y^n\frac{d^nz}{dy^n} = 0.$$

Assume $dx = x\,du$, $dy = y\,dv$; then by Ex. (6) of Chap. III. Sect. 1, of the Diff. Calc. we have generally

$$x^r y^s \frac{d^{r+s}z}{dx^r dy^s} = \frac{d}{du}\left(\frac{d}{du} - 1\right) \dots \left\{\frac{d}{du} - (r-1)\right\} \times$$
$$\frac{d}{dv}\left(\frac{d}{dv} - 1\right) \dots \left\{\frac{d}{dv} - (s-1)\right\}$$

Now if we put for shortness

$$\frac{d}{du}\left(\frac{d}{du}-1\right)\ldots\left\{\frac{d}{du}-(r-1)\right\}=\left[\frac{d}{du}\right]^r,$$

the given equation takes the form

$$\left\{\left[\frac{d}{du}\right]^n+n\left[\frac{d}{du}\right]^{n-1}\left[\frac{d}{dv}\right]+\frac{n(n-1)}{1\,.\,2}\left[\frac{d}{du}\right]^{n-2}\left[\frac{d}{dv}\right]^2+\&c.\right.$$
$$\left.+\left[\frac{d}{dv}\right]^n\right\}z=0.$$

But by a known theorem of Vandermonde if

$$[x]^r=x(x-1)\ldots(x-r+1),$$
$$[x]^n+n[x]^{n-1}[y]+\frac{n(n-1)}{1\,.\,2}[x]^{n-2}[y]^2+\&c.+[y]^n=[x+y]^n.$$

Therefore, as the symbols of differentiation are subject to the same laws of combination as the algebraical symbols, the differential equation may be written

$$\left[\frac{d}{du}+\frac{d}{dv}\right]^n z=0;$$

or $\left(\frac{d}{du}+\frac{d}{dv}\right)\left(\frac{d}{du}+\frac{d}{dv}-1\right)\ldots\left\{\frac{d}{du}+\frac{d}{dv}-(n-1)\right\}z=0;$

the integral of which is

$$z=\phi_0(v-u)+\epsilon^u\phi_1(v-u)+\&c.+\epsilon^{(n-1)u}\phi_{n-1}(v-u);$$

or $z=f_0\left(\frac{y}{x}\right)+xf_1\left(\frac{y}{x}\right)+x^2f_2\left(\frac{y}{x}\right)+\&c.+x^{n-1}f_{n-1}\left(\frac{y}{x}\right);$

f_0, f_1, &c. being arbitrary functions.

(15) Let $xy\frac{d^2z}{dx\,dy}+ax\frac{dz}{dx}+by\frac{dz}{dy}\,abz=V,$

(V being a function of x and y).

Putting as before $\frac{dx}{x}=du,\ \frac{dy}{y}=dv$, this becomes

$$\left(\frac{d}{du}+b\right)\left(\frac{d}{dv}+a\right)z=V;$$

the integral of which (see Ex. (11) of the preceding section) is

$$z = \epsilon^{-(av+bu)} \int dv\, \epsilon^{av} \int du\, \epsilon^{bu}\, V + \epsilon^{-av} \phi(u) + \epsilon^{-bu} \psi(v);$$

$$\text{or}\quad z = \frac{1}{x^b y^a} \int dy\, y^{a-1} \int dx\, x^{b-1}\, V + \frac{1}{y^a} f(x) + \frac{1}{x^b} F(y).$$

(16) Let $x^2 \dfrac{d^2 z}{dx^2} - y^2 \dfrac{d^2 z}{dy^2} + x \dfrac{dz}{dx} - y \dfrac{dz}{dy} = 0.$

By the same transformation as before we find

$$z = \phi\left(\frac{y}{x}\right) + \psi(xy).$$

(17) Let $\dfrac{d^2 z}{dx^2} + \dfrac{2}{x} \dfrac{dz}{dx} = a^2 \dfrac{d^2 z}{dy^2}.$

By the same process as in Ex. (9) of Chap. IV. Sect. 2, this may be put under the form

$$\frac{d^2 (xz)}{dx^2} - a^2 \frac{d^2 (xz)}{dy^2} = 0.$$

Whence we find

$$z = \frac{1}{x} \{\phi(y + ax) + \psi(y - ax)\}.$$

(18) Let $\dfrac{d^2 z}{dy^2} = a^2 \left(\dfrac{d^2 z}{dx^2} + \dfrac{2}{x} \dfrac{dz}{dx} - \dfrac{2}{x^2} z\right).$

This may be put under the form

$$\frac{d^2 z}{dy^2} = a^2 \frac{d}{dx} \left(\frac{d}{dx} + \frac{2}{x}\right) z,$$

and thence by the same process as in Ex. (10) of Chap. IV. Sect. 2, we find

$$\frac{d^2 v}{dy^2} = a^2 \frac{d^2 v}{dx^2}, \text{ where } v = x \left(\frac{d}{dx}\right)^{-1} z.$$

Integrating we have

$$v = \phi(x + ay) + \psi(x - ay);$$

and therefore

$$z=\frac{1}{x}\left\{\phi'(x+ay)+\psi'(x-ay)\right\}-\frac{1}{x^2}\left\{\phi(x+ay)+\psi(x-ay)\right\}.$$

This equation occurs in the Theory of Sound. See Airy's *Tracts*, p. 271.

(19) Let $\dfrac{d^2z}{dx^2}-\dfrac{a^2}{x^4}\dfrac{d^2z}{dy^2}=0.$

This equation is of the same form as that in Ex. (6) of Chap. v., and its integral will be found from that given there by putting $a\dfrac{d}{dy}$ for c, and changing the arbitrary constants into arbitrary functions of y. Hence we find

$$z=x\left\{F\left(y+\frac{a}{x}\right)+f\left(y-\frac{a}{x}\right)\right\}.$$

(20) The integral of the equation

$$\frac{d^2z}{dx^2}-\frac{a^2}{x^{\frac{4}{3}}}\frac{d^2z}{dy^2}=0,$$

may in the same way be deduced from that of Ex. (8) of the same Chapter: the result is

$$Z=x\left\{F'(y+3ax^{\frac{1}{3}})+f'(y-3ax^{\frac{1}{3}})\right\}$$
$$-\frac{1}{3a}\left\{F(y+3ax^{\frac{1}{3}})-f(y-3ax^{\frac{1}{3}})\right\}.$$

(21) Let $\dfrac{d^2z}{dx^2}-a^2\dfrac{d^2z}{dy^2}=\dfrac{2z}{x^2}.$

The integral of this equation may be deduced from that in Ex. (10) of Chap. v. by putting $-a^2\dfrac{d^2}{dy^2}$ for q^2. This gives us

$$z=\frac{1}{ax}\left\{F(y-ax)-f(y+ax)\right\}+F'(y-ax)+f'(y+ax)$$

(22) Let $\dfrac{d^2z}{dx^2}+a\dfrac{d^2z}{dx\,dy}=\dfrac{2z}{x^2}.$

The integral of this equation is deduced from that in Ex. (13) of Chap. v., by putting $a\dfrac{d}{dy}$ for q. This gives

$$z = aF'(y) - \frac{2}{x}F(y) + af'(y - ax) + \frac{2}{x}f(y - ax).$$

(23) The equation

$$\frac{d^m z}{dx^m} - \frac{pm}{x}\,\frac{d^{m-1}z}{dx^{m-1}} - a^2\frac{d^2 z}{dy^2} = 0,$$

may be integrated by the same method as that in Ex. (14) of Chap. vi., by changing k^2 into $-a^2\dfrac{d^2}{dy^2}$ and putting arbitrary functions of y instead of the arbitrary constants.

Thus if $m = 2$, we have

$$X = F(y + ax) + f(y - ax),$$

so that the integral of

$$\frac{d^2 z}{dx^2} - \frac{2p}{x}\,\frac{dz}{dx} - a^2\frac{d^2 z}{dy^2} = 0,$$

is $$z = x^{2p+1}\left(\frac{1}{x}\frac{d}{dx}\right)^p \frac{1}{x}\left\{F(y + ax) + f(y - ax)\right\}.$$

Hence if $p = 2$, the integral of

$$\frac{d^2 z}{dx^2} - \frac{4}{x}\,\frac{dz}{dx} - a^2\frac{d^2 z}{dy^2} = 0$$

is $$z = 3\left\{F(y+ax)+f(y-ax)\right\} - 3ax\left\{F'(y+ax)-f'(y-ax)\right\} + a^2x^2\left\{F''(y + ax) + f''(y - ax)\right\}.$$

(24) Let $$\frac{d^2 z}{dx\,dy} + \frac{1}{x + y}\left(\frac{dz}{dx} + \frac{dz}{dy}\right) - \frac{2}{(x + y)^2}z = 0.$$

Assume $y+x = u$, $y - x = v$, when the equation becomes

$$\frac{d^2 z}{dv^2} = \frac{d^2 z}{du^2} + \frac{2}{u}\,\frac{dz}{du} - \frac{2}{u^2}z = 0.$$

The integral of this by Ex. (18) is

$$z = \frac{1}{u}\left\{\phi'(u + v) + \psi'(u - v)\right\} - \frac{1}{u^2}\left\{\phi(u + v) + \psi(u - v)\right\}.$$

Hence,

$$z = \frac{1}{x+y}\{\phi'(2y) + \psi'(2x)\} - \frac{1}{(x+y)^2}\{\phi(2y) + \psi(2x)\}.$$

Non-linear equations of the form

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = \phi(z) \cdot R,$$

P, Q, R being functions of x and y, may be transformed into linear equations by assuming

$$\frac{dz}{\phi(z)} = dz'.$$

(25) Let $x\dfrac{dz}{dx} + y\dfrac{dz}{dy} = \dfrac{xy}{z}$.

The transformed equation is, putting

$$z\,dz = dz', \text{ or } z' = \frac{z^2}{2},$$

$$x\frac{dz'}{dx} + y\frac{dz'}{dy} = xy;$$

and the integral is

$$z^2 = xy + \phi\left(\frac{x}{y}\right).$$

(26) Let $x^2\dfrac{dz}{dx} + y^2\dfrac{dz}{dy} = \dfrac{(1+z^2)^{\frac{1}{2}}}{z}$.

By assuming $\dfrac{z\,dz}{(1+z^2)^{\frac{1}{2}}} = dz'$ or $z' = (1+z^2)^{\frac{1}{2}}$, this becomes

$$x^2\frac{dz'}{dx} + y^2\frac{dz'}{dy} = 1,$$

and the integral is

$$(1+z^2)^{\frac{1}{2}} = -\frac{1}{x} + \phi\left(\frac{1}{y} - \frac{1}{x}\right).$$

(27) Let $x\dfrac{dz}{dx} + y\dfrac{dz}{dy} = 2xy\,(a^2 - z^2)^{\frac{1}{2}}$.

The integral is

$$z = a\sin\left\{xy + \phi\left(\frac{y}{x}\right)\right\}.$$

We might with advantage have applied the same transformation to the equations in examples (1), (3), and (4), as it is generally convenient to reduce the factor of z to two terms.

SECT. 3. *Equations involving the differential coefficients of z in powers and products.*

If the equation be of the first order make $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, and from the given equation find q in terms of p, x, y, z, and substitute this value in the equation

$$\frac{dp}{dy} - \frac{dq}{dx} + q\frac{dp}{dz} - p\frac{dq}{dz} = 0, \qquad (1)$$

which will then become an equation of the first order between four variables. The value of p found by integrating this, with the corresponding value of q will render

$$dz = p\,dx + q\,dy, \qquad (2)$$

a complete differential, and this being integrated will give the value of z. The integral of the first equation will involve an arbitrary constant (a); and the integral of the second will introduce another (b), which is to be considered as an arbitrary function of (a); and we shall thus obtain an integral of the form

$$f(x, y, z, a) = \phi(a),$$

from which a is to be eliminated when a specific meaning is assigned to ϕ.

Lagrange, *Mémoires de Berlin*, 1772, p. 353.

(1) Let $p^2 + q^2 = 1$, or $q = (1 - p^2)^{\frac{1}{2}}$,

$$\frac{dq}{dx} = -\frac{p}{(1 - p^2)^{\frac{1}{2}}}\frac{dp}{dx}, \quad \frac{dq}{dz} = -\frac{p}{(1 - p^2)^{\frac{1}{2}}}\frac{dp}{dz}.$$

Substituting these values in equation (1) it becomes

$$\frac{dp}{dz} + p\frac{dp}{dx} + (1 - p^2)^{\frac{1}{2}}\frac{dp}{dy} = 0.$$

This equation is integrable if we can integrate the system of equations

$$dp = 0, \quad p\,dz - dx = 0, \quad (1 - p^2)^{\frac{1}{2}}dz - dy = 0.$$

The first gives $p = a$, whence $q = (1 - a^2)^{\frac{1}{2}}$, and

$$dz = a\,dx + (1 - a^2)^{\frac{1}{2}}\,dy,$$

so that
$$z = ax + (1 - a^2)^{\frac{1}{2}}\,y + \phi(a).$$

If we differentiate this with respect to a we obtain the equation

$$0 = x - \frac{a}{(1 - a^2)^{\frac{1}{2}}}\,y + \phi'(a),$$

between which and the preceding we may eliminate a when ϕ is specified.

(2) Let
$$pq = 1.$$

The equation in p to be integrated is

$$\frac{dp}{dy} + \frac{1}{p^2}\frac{dp}{dx} + \frac{2}{p}\frac{dp}{dz} = 0;$$

whence $dp = 0$ and $p = a$. The final integral is

$$p = ax + \frac{y}{a} + \phi(a).$$

(3) Let
$$z = pq.$$

In this case we find

$$p = y + a, \quad q = \frac{z}{y + a};$$

$$\text{therefore} \quad dz = (y + a)\,dx + \frac{z}{y + a}\,dy,$$

$$\text{whence} \quad \frac{dz}{y + a} - \frac{z\,dy}{(y + a)^2} = dx;$$

$$\text{and therefore} \quad \frac{z}{y + a} = x + \phi(a).$$

(4) Let
$$p = (qy + z)^2.$$

In this case we get

$$qy^2 = a, \quad \text{and} \quad p = \left(\frac{a}{y} + z\right)^2,$$

$$\text{whence} \quad dz = \left(z + \frac{a}{y}\right)^2 dx + \frac{a}{y^2}\,dy;$$

$$\text{or} \quad d\left(z + \frac{a}{y}\right) = \left(z + \frac{a}{y}\right)^2 dx,$$

$$\text{whence} \quad \left(z + \frac{a}{y}\right)\{x + \phi(a)\} + 1 = 0.$$

(5) Let $$q = p^2 z,$$

Here $p = \dfrac{a}{z}$, $q = \dfrac{a^2}{z}$, and the integral is

$$\frac{z^2}{2} = ax + a^2 y + \phi(a).$$

(6) Let $$q = xp + p^2.$$

The integral is

$$z = x\epsilon^{y+a} + \tfrac{1}{2}\epsilon^{2(y+a)} + \phi(a).$$

(7) Let $x^\alpha y^\beta z^\gamma \left(\dfrac{dz}{dx}\right)^m \left(\dfrac{dz}{dy}\right)^n = c^n.$

This may be put under the form

$$\left(x^{\frac{\alpha}{m}} z^{\frac{\gamma}{m+n}} \frac{dz}{dx}\right)^m \left(y^{\frac{\beta}{n}} z^{\frac{\gamma}{m+n}} \frac{dz}{dy}\right)^n = c^n.$$

By assuming

$$x' = \int x^{-\frac{\alpha}{m}} dx, \quad y' = \int y^{-\frac{\beta}{n}} dy, \quad z' = \int z^{\frac{\gamma}{m+n}} dz;$$

the equation becomes

$$\left(\frac{dz'}{dx'}\right)^m \left(\frac{dz'}{dy'}\right)^n = c^n.$$

The integral of this found by the same method as in Ex. (2) is

$$z' = a^n x' + \frac{c}{a^m} y' + \phi(a);$$

and therefore

$$\frac{m+n}{m+n+\gamma} z^{\frac{m+n+\gamma}{m+n}} = \frac{m}{m-\alpha} a^n x^{\frac{m-\alpha}{m}} + \frac{n}{n-\beta} \frac{1}{a^m} y^{\frac{n-\beta}{n}} + \phi(a).$$

When $m = \alpha$, $x' = \log x$, when $n = \beta$, $y' = \log y$,

when $m + n + \gamma = 0$, $z' = \log z$.

(8) This transformation fails when $m+n=0$ while γ is not equal to 0. In this case the following method may be used. The equation may evidently be put under the form

$$\left(\frac{dz}{dx}\right)^m \left(\frac{dz}{dy}\right)^n z^\gamma = c^n;$$

then considering x as a function of z and y,

$$dx = \frac{dx}{dz}dz + \frac{dx}{dy}dy,$$

and therefore
$$dz = \frac{1}{\dfrac{dx}{dz}}dx - \frac{\dfrac{dx}{dy}}{\dfrac{dx}{dz}}dy;$$

whence
$$\frac{dz}{dx} = \frac{1}{\dfrac{dx}{dz}},\quad \frac{dz}{dy} = -\frac{\dfrac{dx}{dy}}{\dfrac{dx}{dz}}.$$

By substituting these values the equation becomes (since $m+n=0$)

$$\left(\frac{dx}{dy}\right)^n z^\gamma = (-)^n c^n;$$

the integral of which is

$$x = -cyz^{-\frac{\gamma}{n}} + \phi(z).$$

(9) Let $\left(\dfrac{dz}{dy}\right)^2 xz = \left(\dfrac{dz}{dx}\right)^2 y^2.$

The transformed equation is

$$\left(\frac{dx'}{dy'}\right)^2 z = -1,$$

and the integral is

$$\frac{2}{3}x^{\frac{3}{2}} + \frac{y^2}{2z^{\frac{1}{2}}} = f(z).$$

The transformations in the three preceding examples are given by a writer who signs himself "G. C." in the *Cambridge Mathematical Journal*, Vol. I. p. 162.

SECT. 4. *Equations integrable by various methods.*

Lagrange's Method.

Let a partial Differential Equation between three variables be of the form

$$P\frac{dz}{dx} + Q\frac{dz}{dy} = R,$$

where P, Q, R are functions of x, y and z; then if we can integrate two of the following equations,

$$Pdy - Qdx = 0$$

$$Pdz - Rdx = 0$$

$$Qdz - Rdy = 0,$$

so as to obtain two integrals,

$$\phi(x, y, z) = \beta, \quad \psi(x, y, z) = a,$$

the integral of the given equation will be

$$\beta = f(a).$$

Lagrange, *Mémoires de Berlin*, 1774, p. 197; 1779, p. 152.

For the success of this method it is necessary either that one of the three auxiliary equations should contain only the two variables the differentials of which it involves, or that by their combination such an equation should be obtained. By integrating it we obtain an equation by means of which one of the variables may be eliminated from either of the other auxiliary equations.

(1) Let $$x\frac{dz}{dx} + z\frac{dz}{dy} + y = 0.$$

In this case the auxiliary equations are

$$xdy - zdx = 0,$$

$$xdz + ydx = 0,$$

$$zdz + ydy = 0.$$

The last of these alone is immediately integrable and gives

$$z^2 + y^2 = a^2.$$

Substituting in the second, we have

$$\frac{dz}{(a^2 - z^2)^{\frac{1}{2}}} + \frac{dx}{x} = 0,$$

whence $x e^{\sin^{-1}\frac{z}{a}} = b$, and therefore

$$x e^{\sin^{-1}\frac{z}{a}} = f(y^2 + x^2),$$

is the integral of the proposed equation.

(2) Let $\quad y^3 \dfrac{dz}{dy} - xy^2 \dfrac{dz}{dx} = axz;$

$$\text{then} \quad \log z = -\frac{ax}{3y^2} + f(xy).$$

(3) Let $\quad (y - bz) \dfrac{dz}{dx} - (x - az) \dfrac{dz}{dy} = bx - ay.$

The auxiliary equations are,

$$(y - bz)\, dy + (x - az)\, dx = 0 \,\ldots\ldots\ldots\, (1)$$

$$(y - bz)\, dz - (bx - ay)\, dx = 0 \,\ldots\ldots\ldots\, (2)$$

$$(x - az)\, dz + (bx - ay)\, dy = 0 \,\ldots\ldots\ldots\, (3).$$

Multiply (2) by a, and (3) by b; subtract and divide by $bx - ay$: we find

$$dz + a\,dx + b\,dy = 0, \quad \text{or} \quad z + ax + by = \alpha.$$

Again multiply (2) by x, and (3) by y; subtract and divide by $bx - ay$: there results

$$x\,dx + y\,dy + z\,dz = 0, \text{ whence } x^2 + y^2 + z^2 = \beta.$$

Therefore

$$x^2 + y^2 + z^2 = f(z + ax + by)$$

is the integral of the proposed equation. This is the general equation to surfaces of revolution.

(4) Let $\quad x^2 \dfrac{dz}{dx} + y^2 \dfrac{dz}{dy} = nxy,$

$$z = \frac{nxy}{y - x} \log \frac{y}{x} + f\left(\frac{x - y}{xy}\right).$$

(5) Let $$y^2\frac{dz}{dx}+xy\frac{dz}{dy}=nxz;$$

then $$z=y^n f(x^2-y^2).$$

(6) Let $$(x+y)\frac{dz}{dx}+(y-x)\frac{dz}{dy}=z.$$

The auxiliary equations are,

$$(x+y)\,dy-(y-x)\,dx=0 \;\ldots\ldots\ldots\ldots\; (1)$$

$$(x+y)\,dz-z\,dx \quad =0 \;\ldots\ldots\ldots\ldots\; (2)$$

$$(y-x)\,dz-z\,dy \quad =0 \;\ldots\ldots\ldots\ldots\; (3).$$

Equation (1) may be put under the form

$$x\,dy-y\,dx+x\,dx+y\,dy=0;$$

whence $$\tan^{-1}\frac{x}{y}-\log(x^2+y^2)^{\frac{1}{2}}=a.$$

Multiplying (2) by x, (3) by y and adding, we have

$$\frac{dz}{z}=\frac{x\,dx+y\,dy}{x^2+y^2},$$

whence $$\frac{z}{(x^2+y^2)^{\frac{1}{2}}}=\beta;$$ and therefore

$$z=(x^2+y^2)^{\frac{1}{2}}f\left\{\tan^{-1}\frac{x}{y}-\log(x^2+y^2)^{\frac{1}{2}}\right\}$$

is the required integral.

(7) Let $$(x-2y)\frac{dz}{dx}+(2x-3y)\frac{dz}{dy}=z.$$

The integral is

$$(x-y)\,z=e^{\frac{x}{x-y}}f(x-y)^2.$$

This method may be extended to functions of more variables. Thus if

$$P\frac{du}{dx}+Q\frac{du}{dy}+R\frac{du}{dz}=S,$$

and if from three equations such as

$$Pdy - Qdx = 0,$$
$$Pdz - Rdx = 0,$$
$$Pdu - Sdx = 0,$$

we can obtain three integrals,

$$U = a, \quad V = b, \quad W = c,$$

the integral of the proposed equation is

$$U = f(V, W), \text{ or } \phi(U, V, W) = 0.$$

(8) Let

$$(u+y+z)\frac{du}{dx} + (u+x+z)\frac{du}{dy} + (u+x+y)\frac{du}{dz} = x+y+z.$$

The auxiliary equations are,

$$(u+y+z)\,dy - (u+x+z)\,dx = 0,$$
$$(u+y+z)\,dz - (u+x+y)\,dx = 0,$$
$$(u+y+z)\,du - (x+y+z)\,dx = 0.$$

Adding these three equations we have

$$(u+y+z)(du + dx + dy + dz) - 3(u+x+y+z)\,dx = 0.$$

Putting $u + x + y + z = v$, this gives

$$\frac{dx}{u+y+z} = \frac{dv}{3v}.$$

Subtracting the second equation from the first, we have

$$(u+y+z)(dy - dz) = (z - y)\,dx;$$

$$\text{or} \quad \frac{dx}{u+y+z} = -\frac{dy - dz}{y - z}.$$

Therefore $$\frac{dv}{3v} = -\frac{dy - dz}{y - z};$$

and $v(y - z)^3 = a.$

From the symmetry of the expressions it is obvious that we must have also

$$v(x - z)^3 = b, \qquad v(u - z)^3 = c.$$

Therefore

$$f\left\{v(u - z)^3,\ v(x - z)^3,\ v(y - z)^3\right\} = 0.$$

(9) Let $x\frac{du}{dx} + (u+z)\frac{du}{dy} + (u+y)\frac{du}{dz} = y+z.$

The integral of this is

$$u + x + y + z = y^2 f\{x(u-y),\ x(y-z)\}.$$

Monge's Method.

Let the partial differential equation be of the form

$$\frac{d^2z}{dx^2} + P\frac{d^2z}{dx\,dy} + Q\frac{d^2z}{dy^2} = R,$$

where P, Q, R are functions of x, y, z, $\frac{dz}{dx}$, $\frac{dz}{dy}$.

Then if we form the system of equations

$$\left.\begin{aligned} dy - m\,dx &= 0 \\ m\,dp + Q\,dq - Rm\,dx &= 0 \end{aligned}\right\}\ldots\ldots(1)$$

$$\left.\begin{aligned} dy - m'dx &= 0 \\ m'dp + Q\,dq - Rm'dx &= 0 \end{aligned}\right\}\ldots\ldots(2)$$

where $p = \frac{dz}{dx}$ and $q = \frac{dz}{dy}$, and m, m' are the roots of the equation

$$m^2 - Pm + Q = 0;$$

and if from these two systems we can find two integrals $U = a$, $V = b$, then

$$V = f(U)$$

is the first integral of the proposed equation; and the integral of this is the complete integral of the proposed equation. It is generally more convenient (*when possible*) to find another first integral, of the form

$$V' = f_{\prime}(U'),$$

and between these to eliminate p or q so as to obtain an equation involving only one differential coefficient, and which is therefore easily integrable.

Monge, *Mémoires de l'Académie des Sciences*, 1784, p. 118.

(10) Let $q^2\frac{d^2z}{dx^2} - 2pq\frac{d^2z}{dx\,dy} + p^2\frac{d^2z}{dy^2} = 0.$

The auxiliary equations in this case are,

$$q^2m^2 + 2pqm + p^2 = 0, \quad \text{whence } m = -\frac{p}{q},$$

$$qdy + pdx = 0,$$

$$-pdp + \frac{p^2}{q}\,dq = 0.$$

From the second, since $dz = pdx + qdy$, we have

$$dz = 0, \quad \text{or } z = a.$$

From the third we have

$$pdq - qdp = 0;$$

$$\text{whence } \frac{p}{q} = \phi(z) = c,$$

since z = constant, and therefore $\phi(z)$ = constant. From the equation $p - cq = 0$, we easily obtain

$$z = f(x + cy) = f\{x + y\phi(z)\},$$

which is the required integral.

(11) Let $$\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} + \frac{4p}{x+y} = 0.$$

The auxiliary equations are,

$$dy - dx = 0, \quad dp - dq + \frac{4p}{x+y}\,dx = 0 \;\ldots\ldots\; (1)$$

$$dy + dx = 0, \quad dp + dq - \frac{4p}{x+y}\,dx = 0 \;\ldots\ldots\; (2)$$

From the first of (1) we find

$$y - x = c, \quad \text{and therefore } dp - dq + \frac{4p\,dx}{2y-a} = 0.$$

If we subtract from this last the equation

$$\frac{2p}{2y-a}(dy - dx) = \frac{2pdy}{2y-a} - \frac{2dz}{2y-a} + \frac{2qdy}{2y-a}$$

(as $pdx = dz - qdy$) we have

$$(2y - a)(dp - dq) + 2(p - q)\,dy + 2dz = 0;$$

the integral of which is

$$(2y-a)(p-q)+2z=b=f(y-x),$$

and therefore

$$p-q+\frac{2z}{x+y}=\frac{f(y-x)}{x+y}.$$

From the first of (2) we find

$$y+x=a_1,$$

and substituting this in the equation just found, it becomes

$$\frac{dz}{dy}-\frac{dz}{dx}-\frac{2z}{a_1}=-\frac{f(y-x)}{a_1}.$$

This is a linear equation and is therefore easily integrated. The result is

$$z=-\epsilon^{\frac{2y}{x+y}}\int dy\,\epsilon^{-\frac{2y}{a_1}}\frac{f(y-x)}{a_1}+\epsilon^{\frac{2y}{x+y}}\psi(x+y),$$

where $x+y$ is to be substituted for a_1 *after* integration.

(12) Let the equation be

$$(1+pq+q^2)\frac{d^2z}{dx^2}+(q^2-p^2)\frac{d^2z}{dx\,dy}-(1+pq+p^2)\frac{d^2z}{dy^2}=0.$$

If we put $p+q=a$, this takes the form

$$(1+qa)\frac{d^2z}{dx^2}+(q-p)\,a\frac{d^2z}{dx\,dy}-(1+pa)\frac{d^2z}{dy^2}=0.$$

The equation for determining m is

$$(1+qa)m^2-(q-p)\,am-(1+pa)=0;$$

which gives $\quad m=1, \quad m=-\dfrac{1+pa}{1+qa}.$

We have therefore to integrate the two systems,

$$dy-dx=0;\quad dp(1+qa)-dq(1+pa)=0\ \dots\ (1),$$

$$dy(1+qa)+dx(1+pa)=0;\quad dp+dq=0\ \dots\ (2).$$

The second equation of (2), gives $p+q=b$ or $a=b$. The first equation, of (2) when put under the form

$$dx+dy+a(p\,dx+q\,dy)=0,$$

gives $$x + y + (p + q)z = a;$$
therefore $$x + y + (p + q)z = \phi(p + q).$$

The first equation of (1) gives $p + q = a$, and putting $p - q = \beta$, we have

$$p = \tfrac{1}{2}(a + \beta),\ q = \tfrac{1}{2}(a - \beta),$$
$$dp = \tfrac{1}{2}(da + d\beta),\ dq = \tfrac{1}{2}(da - d\beta),$$

and therefore the second equation of (1) may be put under the form

$$\frac{d\beta}{\beta} = \frac{a\,da}{2 + a^2};\ \text{whence } \beta = b_1(2 + a^2)^{\frac{1}{2}},$$

and therefore

$$p - q = \psi(x - y)\{2 + (p + q)^2\}^{\frac{1}{2}}.$$

This first integral will enable us to determine the second integral. Putting $p + q = a$, $p - q = \beta$, we have

$$dz = \tfrac{1}{2}(a + \beta)dx + \tfrac{1}{2}(a - \beta)dy = \tfrac{1}{2}a(dx + dy) + \tfrac{1}{2}\beta(dx - dy);$$

or, putting for β its value $\psi(x - y)(2 + a^2)^{\frac{1}{2}}$,

$$dz = \tfrac{1}{2}a(dx + dy) + \tfrac{1}{2}(dx - dy)\,\psi(x - y)(2 + a^2)^{\frac{1}{2}}.$$

This is integrable if we suppose a to be constant, and gives

$$z + \phi(a) = \tfrac{1}{2}a(x + y) + \psi_1(x - y)(2 + a^2)^{\frac{1}{2}};$$

which, combined with

$$\phi'(a) = \tfrac{1}{2}(x + y) + \psi_1(x - y)\frac{a}{(2 + a^2)^{\frac{1}{2}}},$$

represents the integral of the proposed equation.

Poisson* has shewn how to obtain a particular integral of equations of the form

$$P = (rt - s^2)^n\, Q \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where P is a function of p, q, r, s, t, homogeneous with respect to the last three quantities, and Q is a function of x, y, z, and the differentials of z, which does not become infinite when $rt - s^2 = 0$.

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* *Correspondance sur l'Ecole Polytechnique*, Vol. II. p. 410.

If we assume $q = f(p)$, we have

$$s = rf'(p), \quad t = sf'(p) = r\{f'(p)\}^2;$$

and therefore $rt - s^2 = 0.$ .... (2)

Hence the equation (1) is reduced to

$$P = 0;$$

and on substituting in it the values of q, s, and t, the quantity r will divide out, as P is homogeneous in r, s, and t, and the equation is reduced to the form

$$F\{p, f(p)\, f'(p)\} = 0,$$

which is an ordinary differential equation, and being integrated determines the form of $f(p)$ involving an arbitrary constant. The partial differential equation

$$q = f(p)$$

can always be integrated, and furnishes a value of z involving an arbitrary function and an arbitrary constant. This process comes to the same as finding what developable surfaces satisfy the equation (1).

(13) Let $\qquad r^2 - t^2 = rt - s^2.$

Assuming $q = f(p)$ we find

$$r^2\{1 - [f'(p)]^2\} = 0,$$

whence $\qquad f'(p) = \pm 1;$

and therefore $\qquad q = f(p) = \pm p + C,$

C being an arbitrary constant. On integrating this we find

$$z = Cx \pm \phi(y \pm x)$$

as a particular integral of the given equation.

(14) Let $\quad t + 2ps + (p^2 - a^2)\, r = 0.$ .... (1)

In this case $Q = 0$, and on putting $q = f(p)$ we have, after dividing by r,

$$\{f'(p)\}^2 + 2pf'(p) + p^2 - a^2 = 0; \text{.........(2)}$$

from which $\qquad f'(p) + p = \pm a,$

and therefore $\qquad q + \tfrac{1}{2}p^2 \pm ap = C.$ ....(3)

Now as every equation involving only p and q may be considered as representing a developable surface, it may be satisfied by the equation to a plane in which the arbitrary constants are afterwards supposed to vary. Hence assuming

$$z = \alpha x + \beta y + \gamma,$$

we find $p = \alpha$, $q = \beta$, and therefore

$$\beta = C \pm a\alpha - \tfrac{1}{2}\alpha^2;$$

so that a particular integral of (3) is

$$z = \alpha x + (C \pm a\alpha - \tfrac{1}{2}\alpha^2)\, y + \gamma.$$

To deduce the general integral we must take for γ an arbitrary function of α, and then join with the equation to the plane its differential with respect to α, so that the system of equations

$$z = \alpha x + (C \pm a\alpha - \tfrac{1}{2}\alpha^2)\, y + \phi(\alpha),$$

$$0 = x - (\alpha \pm a)\, y + \phi'(\alpha),$$

is the general integral of (3), and a particular integral of (1). A different form of ϕ should be taken for each sign of a, so that this system is equivalent to two.

The equation

$$(15) \qquad (1 + q^2)\, r - 2pqs + (1 + p^2)\, t = 0,$$

belongs to those surfaces in which the principal radii of curvature are equal but of opposite signs. On assuming $q = f(p)$, we have

$$1 + \{f(p)\}^2 - 2pf(p)\, f'(p) + (1 + p^2)\, \{f'(p)\}^2 = 0.$$

The integral of this is

$$q = ap + (-1 - a^2)^{\frac{1}{2}};$$

from which we have

$$z = \phi(x + ay) + y\, (-1 - a^2)^{\frac{1}{2}}$$

as the particular integral of the given equation.

It is easy to see that this must represent a plane, as that is the only developable surface which has its principal radii of curvature equal and of opposite signs.

From the difficulties attending the integration of ordinary differential equations of a high order it will readily be understood that the integration of partial differential equations of the second and higher orders is a problem in the solution of which still less progress has been made. The subject has much occupied the attention of mathematicians, and processes have been given for integrating various classes of these equations, but they are unfortunately exceedingly long and complex, and the solutions are frequently given in a form which renders them practically useless. I shall therefore not give any examples of them here, but shall content myself with referring the reader to the original memoirs: such as those of Laplace, *Mémoires de l'Académie*, 1773; Legendre, *Ib.* 1787; Ampère, *Journal Polytechnique*, Cahiers XVII. et XVIII.; and Cardinali, *Sul Calcolo Integrale dell' equazioni di differenze partiali.*

Some examples of the application of Definite integrals to express the integrals of partial Differential equations will be found at the end of Chap. XII.

CHAPTER VII.

SIMULTANEOUS DIFFERENTIAL EQUATIONS.

SECT. 1. *Linear Differential Equations with constant Coefficients.*

The solution of any number of simultaneous equations of this class may always be reduced to the principles of the elimination of the same number of linear algebraical equations. For the symbol of differentiation may be treated exactly like any constant involved in the equation, and therefore the rules for eliminating, when the variables are involved along with constants, may be applied to equations in which they are involved, along with symbols of differentiation.

Ex. (1) Let there be two simultaneous equations involving two variables,

$$\frac{dx}{dt} + ay = 0, \qquad \frac{dy}{dt} + bx = 0.$$

To eliminate y, operate on the first equation with $\frac{d}{dt}$ and multiply the second by a; we have then

$$\frac{d^2x}{dt^2} + a\frac{dy}{dt} = 0, \qquad a\frac{dy}{dt} + abx = 0.$$

Subtracting the second of these from the first, y disappears, and we have

$$\frac{d^2x}{dt^2} - abx = 0.$$

The integral of this is, making $ab = m^2$,

$$x = C\epsilon^{mt} + C_1\epsilon^{-mt}.$$

From the first equation we find

$$y = C_1\left(\frac{b}{a}\right)^{\frac{1}{2}}\epsilon^{-mt} - C\left(\frac{b}{a}\right)^{\frac{1}{2}}\epsilon^{mt}.$$

It might at first appear that as we might obtain an equation involving y alone, similar to the resulting one in x, there must be four arbitrary constants, and not two. But the second pair can always be determined in terms of the other two, and are therefore not arbitrary. This remark applies to such equations generally: and it is best to avoid the introduction of the superfluous constants by deducing (as we have done in this example) the other variables from the first without integration. The real number of arbitrary constants is always equal to the sum of the highest indices of differentiation in the different equations.

(2) Let
$$\frac{dx}{dt} + ax + by = 0,$$
$$\frac{dy}{dt} + a_1 x + b_1 y = 0,$$
be two simultaneous equations. Operate on the first with $\left(\frac{d}{dt} + b_1\right)$, and multiply the second by b; then, on subtracting, y disappears and we have
$$\left\{\left(\frac{d}{dt} + a\right)\left(\frac{d}{dt} + b_1\right) - a_1 b\right\} x = 0.$$

This may be put under the form
$$\left(\frac{d}{dt} + h\right)\left(\frac{d}{dt} + k\right) x = 0,$$
where h and k are the roots of the equation
$$z^2 - (a + b_1) z + ab_1 - a_1 b = 0.$$

Integrating in the usual way, we find
$$x = C\epsilon^{-ht} + C_1 \epsilon^{-kt};$$
$$y = \frac{h-a}{b} C\epsilon^{-ht} + \frac{k-a}{b} C_1 \epsilon^{-kt}.$$

(3) Let
$$\frac{dx}{dt} + 4x + 3y = t,$$
$$\frac{dy}{dt} + 2x + 5y = \epsilon^t.$$

Eliminating y, we find

$$\left(\frac{d}{dt}+2\right)\left(\frac{d}{dt}+7\right)x = 1 + 5t - 3e^{t}.$$

Whence $x = -\frac{31}{196} + \frac{5}{14}t - \frac{1}{8}e^{t} + C_1 e^{-7t} + C_2 e^{-2t}$,

and $$y = \frac{9}{98} - \frac{1}{7}t + \frac{5}{24}e^{t} - \frac{5}{3}C_1 e^{-7t} - C_2 e^{-2t}.$$

(4) Let $$\frac{dx}{dt} + 5x + y = e^{t};$$

$$\frac{dy}{dt} + 3y - x = e^{2t}.$$

We find

$$x = \frac{4}{25}e^{t} - \frac{1}{36}e^{2t} + (C_1 t + C_0)e^{-4t};$$

$$y = \frac{1}{25}e^{t} + \frac{7}{36}e^{2t} - (C_1 t + C_0 + C_1)e^{-4t}.$$

(5) Let there be three simultaneous equations,

$$\frac{dx}{dt} + by + cz = 0,$$

$$\frac{dy}{dt} + a'x + c'z = 0,$$

$$\frac{dz}{dt} + a''x + b''y = 0.$$

Operate on the first with $\left(\frac{d}{dt}\right)^2 - b''c'$, on the second with $b''c - b\frac{d}{dt}$, and on the third with $bc' - c\frac{d}{dt}$. Then on adding, the terms involving y and z disappear of themselves, and there remains

$$\left\{\left(\frac{d}{dt}\right)^3 - (a'b + a''c + b''c')\frac{d}{dt} + a'b''c + a''bc'\right\}x = 0.$$

The integral of this is easily seen to be

$$x = C_1 e^{\alpha t} + C_2 e^{\beta t} + C_3 e^{\gamma t},$$

α, β, γ being the roots of

$$z^3 - (a'b + a''c + b''c')\,z + a'b''c + a''bc' = 0.$$

The values of y and z are easily derived from that of x.

(6) Let
$$\frac{d^2x}{dt^2} - ay - bx = c,$$
$$\frac{d^2y}{dt^2} - a'y - b'x = c'.$$

Eliminating y by operating on the first with $\left(\frac{d}{dt}\right)^2 - a'$, multiplying the second by a and adding; there results

$$\left\{\left(\frac{d}{dt}\right)^2 - a'\right\}\left\{\left(\frac{d}{dt}\right)^2 - b\right\} x - ab'x = ac' - ca'.$$

This may be put under the form

$$\left\{\left(\frac{d}{dt}\right)^2 + h^2\right\}\left\{\left(\frac{d}{dt}\right)^2 + k^2\right\} x = ac' - ca',$$

where h^2, k^2 are the roots of the equation

$$z^2 + (a' + b)\,z + a'b - ab'.$$

Hence we find

$$x = \frac{ac' - ca'}{a'b - ab'} + C_1 \cos(hx + \alpha) + C_2 \cos(kx + \beta);$$

and
$$y = \frac{b'c - bc'}{a'b - ab'} - \frac{h^2 + b}{a} C_1 \cos(hx + \alpha) - \frac{k^2 + b}{a} C_2 \cos(kx + \beta).$$

(7) Let
$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} - 2\frac{dy}{dt} + x = \cos 2t,$$
$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + \frac{dx}{dt} + 6y + 5x = \sin t.$$

Eliminating y we obtain the equation

$$\left\{\left(\frac{d}{dt}\right)^4 + 5\left(\frac{d}{dt}\right)^2 + 6\right\} x = 2 \cos 2t - 4 \sin 2t - 2 \cos t;$$

or $\left\{\left(\frac{d}{dt}\right)^2 + 2\right\}\left\{\left(\frac{d}{dt}\right)^2 + 3\right\} x = 2\cos 2t - 4\sin 2t - 2\cos t$;

the integral of which is

$x = \cos 2t - 2\sin 2t - \cos t + C_1 \cos(2^{\frac{1}{2}} t + \alpha) + C_2 \cos(3^{\frac{1}{2}} t + \beta)$,

and from this the value of y is easily found.

Take the system of equations.

$$(8)\quad a_0 x + a_1 \frac{dy}{dt} + a_2 \frac{d^2x}{dt^2} + a_3 \frac{d^3y}{dt^3} + \&c. = m \sin nt,$$

$$a_0 y - a_1 \frac{dx}{dt} + a_2 \frac{d^2y}{dt^2} - a_3 \frac{d^3x}{dt^3} - \&c. = m \cos nt.$$

These may be written under the form

$$(a_0 + a_2 d^2 + a_4 d^4 + \&c.) x + (a_1 d + a_3 d^3 + \&c.) y = m \sin nt,$$

$$(a_0 + a_2 d^2 + a_4 d^4 + \&c.) y - (a_1 d + a_3 d^3 + \&c.) x = m \cos nt,$$

where for convenience the differentials of t are omitted.

Eliminating y we have

$$\{(a_0 + a_2 d^2 + a_4 d^4 + \&c.)^2 + (a_1 d + a_3 d^3 + \&c.)^2\} x = m (a_0 + a_1 n - a_2 n^2 - a_3 n^3 + a_4 n^4 + a_5 n^5 - \&c.) \sin nt.$$

It is obvious from the form of this that the complementary function must be of the form

$$\Sigma (A \cos \lambda t + B \sin \lambda t),$$

where all the values are to be assigned to λ which satisfy the equation

$$(a_0 - a_2 \lambda^2 + a_4 \lambda^4 + \&c.)^2 - \lambda^2 (a_1 - a_3 \lambda^2 + \&c.)^2 = 0.$$

Hence we have

$$x = \Sigma (A \cos \lambda t + B \sin \lambda t) +$$

$$\frac{m (a_0 + a_1 n - a_2 n^2 - a_3 n^3 + a_4 n^4 + a_5 n^5 - \&c.)}{(a_0 - a_2 n^2 + a_4 n^4 + \&c.)^2 - n^2 (a_1 - a_3 n^2 + \&c.)^2} \sin nt;$$

or $$x = \Sigma (A \cos \lambda t + B \sin \lambda t) + \frac{m \sin nt}{a_0 - a_1 n - a_2 n^2 + a_3 n^3 + a_4 n^4 - \&c.}.$$

Whence also

$$y = -\Sigma (A \sin \lambda t - B \cos \lambda t) + \frac{m \cos nt}{a_0 - a_1 n - a_2 n^2 + a_3 n^3 + a_4 n^4 - \&c.}.$$

The same method is applicable to linear partial differential equations in which the coefficients are constants. The two symbols of differentiation are to be treated as two independent constants, since they do not affect each other, and are both subject to the laws which regulate the combinations of ordinary algebraical symbols.

(9) Let $$\frac{dz}{dx} + c\frac{du}{dx} + a\frac{dz}{dy} + bz = 0,$$

$$c'\frac{dz}{dx} + \frac{du}{dx} + a\frac{du}{dy} + bu = 0.$$

These may be put under the forms

$$\left(\frac{d}{dx} + a\frac{d}{dy} + b\right) z + c\frac{d}{dx} u = 0,$$

$$\left(\frac{d}{dx} + a\frac{d}{dy} + b\right) u + c'\frac{d}{dx} z = 0.$$

To eliminate u, operate on the first equation with $\frac{d}{dx} + a\frac{d}{dy} + b$, and on the second with $c\frac{d}{dx}$ and subtract: we have then

$$\left\{\left(\frac{d}{dx}\right)^2 + 2\frac{(a\frac{d}{dy} + b)}{1 - cc'}\frac{d}{dx} + \frac{(a\frac{d}{dy} + b)^2}{1 - cc'}\right\} z = 0.$$

If we call $1 - (cc')^{\frac{1}{2}} = \frac{1}{m}$ and $1 + (cc')^{\frac{1}{2}} = \frac{1}{n}$, this may be divided into the two factors

$$\left\{\frac{d}{dx} + m\left(a\frac{d}{dy} + b\right)\right\}\left\{\frac{d}{dx} + n\left(a\frac{d}{dy} + b\right)\right\} z = 0;$$

the integral of which is

$$z = \epsilon^{-mbx}\phi(y - amx) + \epsilon^{-nbx}\psi(y - anx);$$

and from this u can be found.

(10) Let $$\frac{d^2z}{dx\,dy} + a\frac{du}{dy} = 0, \quad \frac{d^2u}{dx\,dy} + c\frac{d^2z}{dx^2} = 0.$$

Eliminating u we find

$$\frac{d^3 z}{dx^2\,dy} - ac\,\frac{d^2 z}{dx^2} = 0;$$

the integral of this equation is

$$z = \phi(y) + x\,\psi(y) + e^{acy}\,\chi(x);$$

whence also u may be determined.

(11) The equations for determining the small disturbances of an elastic medium in three dimensions are

$$\frac{d^2 u}{dt^2} = a^2\,\frac{d}{dx}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

$$\frac{d^2 v}{dt^2} = a^2\,\frac{d}{dy}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right),$$

$$\frac{d^2 w}{dt^2} = a^2\,\frac{d}{dz}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

See Airy's *Tracts*, p. 279, Note.

We might in this case eliminate v and w by a process similar to that used in Ex. (5) of this section; but the following method is more convenient.

Let $$r = a^2\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right).$$

Then $$\left(\frac{d}{dx}\right)^{-1}\frac{d^2 u}{dt^2} = \left(\frac{d}{dy}\right)^{-1}\frac{d^2 v}{dt^2} = \left(\frac{d}{dz}\right)^{-1}\frac{d^2 w}{dt^2} = r,$$

Operate on each of these by $\left(\frac{d}{dx}\right)^2$, $\left(\frac{d}{dy}\right)^2$, $\left(\frac{d}{dz}\right)^2$ respectively, and add: then

$$\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\} r = \frac{d}{dx}\,\frac{d^2 u}{dt^2} + \frac{d}{dy}\,\frac{d^2 v}{dt^2} + \frac{d}{dz}\,\frac{d^2 w}{dt^2}$$

$$= \frac{d^2}{dt^2}\left(\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz}\right)$$

$$= \frac{1}{a^2}\,\frac{d^2 r}{dt^2}.$$

Hence $$\frac{d^2 r}{dt^2} - a^2 \left\{ \left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2 \right\} r = 0.$$

The integral of this is

$$r = \epsilon^{at\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\}^{\frac{1}{2}}} \phi(x,y,z) + \epsilon^{-at\left\{\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2\right\}^{\frac{1}{2}}} \psi(x,y,z).$$

From this r is determined, and hence we can find u, v, w, as

$$u = \left(\frac{d}{dt}\right)^{-2} \frac{dr}{dx}, \quad v = \left(\frac{d}{dt}\right)^{-2} \frac{dr}{dy}, \quad w = \left(\frac{d}{dt}\right)^{-2} \frac{dr}{dz}.$$

When the equations are not linear there is no general method for integrating them; and therefore the means of doing so must be adapted to the particular case under consideration. Two of the more important examples of such equations, which occur in dynamics, are subjoined.

(12) The equations for determining the motion of a particle attracted to a fixed centre of force varying inversely as the square of the distance are

$$\frac{d^2 x}{dt^2} + \frac{\mu x}{r^3} = 0 \quad (1), \qquad \frac{d^2 y}{dt^2} + \frac{\mu y}{r^3} = 0 \quad (2),$$

where $r^2 = x^2 + y^2$.

Multiply the first equation by y and the second by x, and subtract; then

$$x \frac{d^2 y}{dt^2} - y \frac{d^2 x}{dt^2} = 0.$$

Whence, integrating

$$x \frac{dy}{dt} - y \frac{dx}{dt} = c, \qquad (3)$$

c being an arbitrary constant.

Multiply each term of the first equation by the different sides of this equation; then

$$c \frac{d^2 x}{dt^2} = -\frac{\mu}{r^3}\left(x^2 \frac{dy}{dt} - xy \frac{dx}{dt}\right) = -\mu \frac{d}{dt}\left(\frac{y}{r}\right).$$

Integrating, $$-c \frac{dx}{dt} = \mu \frac{y}{r} + a. \qquad (4)$$

Similarly, by means of the second equation,

$$c\frac{dy}{dt} = \mu\frac{x}{r} + b, \qquad (5)$$

a and b being arbitrary constants.

Multiply these equations by y and x respectively and add, then we find

$$\mu r + ay + bx = c^2. \qquad (6)$$

Multiply (1) by $2\frac{dx}{dt}$, (2) by $2\frac{dy}{dt}$ add and integrate;

then $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\left(\frac{\mu}{r} + k\right). \qquad (7)$$

By squaring (3) we find

$$(x^2 + y^2)\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\} - \left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)^2 = c^2;$$

Therefore $$r^2\left(\frac{dr}{dt}\right)^2 = 2\left(\frac{\mu}{r} + k\right)r^2 - c^2, \qquad (8)$$

whence $$t + a = \int\frac{rdr}{\{2(\mu r + kr^2) - c^2\}^{\frac{1}{2}}}. \qquad (9)$$

If we assume $x = r\cos\theta$, $y = r\sin\theta$, equation (3) becomes

$$r^2\frac{d\theta}{dt} = c;$$

whence $$\theta + \beta = \int\frac{cdt}{r^2} = \int\frac{cdr}{r\{2(\mu r + kr^2) - c^2\}^{\frac{1}{2}}}. \qquad (10)$$

From (10) we know θ in terms of r, and from (9) r in terms of $t + a$, so that θ can be expressed in terms of $t + a$, and therefore also x and y in terms of the same quantity. There appear to be five arbitrary constants, a, b, c, a, β, but the equation (6) gives a relation between them which reduces the number of independent constants to four.

(13) Let the equations be

$$a\frac{dx}{dt} + (c - b)yz = 0,$$

$$b\frac{dy}{dt}+(a-c)\,xz=0,$$

$$c\frac{dz}{dt}+(b-a)\,xy=0.$$

These are the equations for determining the angular velocities of a rigid body revolving round its centre of gravity and acted on by no forces.

Multiply the equations by x, y, z, respectively, and let $xyz=\frac{d\phi}{dt}$. Then the first equation gives

$$ax\frac{dx}{dt}+(c-b)\frac{d\phi}{dt}=0.$$

Whence by integration

$$x^2=\frac{2(b-c)}{a}\phi+h^2,$$

h^2 being an arbitrary constant.

Similarly $$y^2=\frac{2(c-a)}{b}\phi+h_1^2,$$

and $$z^2=\frac{2(a-b)}{c}\phi+h_2^2.$$

Hence we find

$$\frac{d\phi}{dt}=xyz$$

$$=\left[\left\{\frac{2(b-c)}{a}\phi+h^2\right\}\left\{\frac{2(c-a)}{b}\phi+h_1^2\right\}\left\{\frac{2(a-b)}{c}\phi+h_2^2\right\}\right]^{\frac{1}{2}}.$$

On inverting and integrating we should obtain t in terms of ϕ, and therefore ϕ in terms of t, and from the value of ϕ, x, y, z in terms of t.

(14) M. Binet* has shewn how to integrate the system of simultaneous equations:

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* *Journal de Mathématiques*, Vol. II. p. 457.

$$\frac{d^2u}{dt^2} = \frac{dR}{du}, \quad \frac{d^2v}{dt^2} = \frac{dR}{dv}, \quad \frac{d^2x}{dt^2} = \frac{dR}{dx}, \text{ \&c.} \qquad (1)$$

the number of variables $u, v, x \ldots$ being n, and R being a function of $r = (u^2 + v^2 + x^2 + \ldots)^{\frac{1}{2}}$, so that

$$\frac{dR}{du} = \frac{dR}{dr}\frac{u}{r}, \quad \frac{dR}{dv} = \frac{dR}{dr}\frac{v}{r}, \quad \frac{dR}{dx} = \frac{dR}{dr}\frac{x}{r}, \text{ \&c.}$$

The equations may therefore be written

$$\frac{d^2u}{dt^2} = \frac{dR}{dr}\frac{u}{r}; \quad \frac{d^2v}{dt^2} = \frac{dR}{dr}\frac{v}{r}; \quad \frac{d^2x}{dt^2} = \frac{dR}{dr}\frac{x}{r}; \text{ \&c.} \qquad (2)$$

Eliminating $\frac{dR}{dr}$ between each pair successively we find equations of the form

$$v\frac{d^2u}{dt^2} - u\frac{d^2v}{dt^2} = 0, \quad x\frac{d^2u}{dt^2} - u\frac{d^2x}{dt^2} = 0, \text{ \&c.} \qquad (3)$$

From these, being $\frac{n(n-1)}{2}$ in number, we obtain the integrals

$$v\frac{du}{dt} - u\frac{dv}{dt} = c_1; \quad x\frac{du}{dt} - u\frac{dx}{dt} = c_2, \text{ \&c.} \qquad (4)$$

The sum of the squares of these gives

$$\left(v\frac{du}{dt} - u\frac{dv}{dt}\right)^2 + \left(x\frac{du}{dt} - u\frac{dx}{dt}\right)^2 + \text{\&c.} = A^2, \qquad (5)$$

where A^2 is the sum of squares of the constants. By adding and subtracting $u^2\left(\frac{du}{dt}\right)^2 + v^2\left(\frac{dv}{dt}\right)^2 + x^2\left(\frac{dx}{dt}\right)^2 + \text{\&c.}$, this may be put under the form

$$(u^2 + v^2 + x^2 + \text{\&c.})\left\{\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \text{\&c.}\right\}$$

$$- \left(u\frac{du}{dt} + v\frac{dv}{dt} + x\frac{dx}{dt} + \text{\&c.}\right)^2 = A^2. \qquad (6)$$

On multiplying the proposed equations by $2\frac{du}{dt}$, $2\frac{dv}{dt}$, $2\frac{dx}{dt}$, &c. and integrating, we have

$$\left(\frac{du}{dt}\right)^2 + \left(\frac{dv}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2 + \text{\&c.} = 2(R+B), \qquad (7)$$

$2B$ being the arbitrary constant arising in the integration.

Substituting this expression in (6) and putting r^2 for $u^2+v^2+x^2+$ &c., that equation becomes

$$\left(u\frac{du}{dt} + v\frac{dv}{dt} + x\frac{dx}{dt} + \text{\&c.}\right)^2 = 2r^2(R+B) - A^2.$$

Now $$u\frac{du}{dt} + v\frac{dv}{dt} + x\frac{dx}{dt} = r\frac{dr}{dt};$$

therefore $$\left(\frac{dr}{dt}\right)^2 = 2(R+B) - \frac{A^2}{r^2}; \qquad (8)$$

and $$\frac{dt}{dr} = \frac{r}{\{2r^2(R+B) - A^2\}^{\frac{1}{2}}}. \qquad (9)$$

By differentiating (8) we find

$$\frac{d^2r}{dt^2} = \frac{dR}{dr} + \frac{A^2}{r^3}. \qquad (10)$$

Eliminating $\frac{dR}{dr}$ from the first of equations (2) by means of (10), and multiplying by r, we have

$$r\frac{d^2u}{dt^2} - u\frac{d^2r}{dt^2} + \frac{A^2}{r^2}\frac{u}{r} = 0,$$

which may be put under the form

$$\frac{d}{dt}\left\{r^2\frac{d}{dt}\left(\frac{u}{r}\right)\right\} + \frac{A^2}{r^2}\frac{u}{r} = 0;$$

or by multiplying by $\frac{r^2}{A^2}$, and assuming

$$d\phi = \frac{A\,dt}{r^2} = \frac{A\,dr}{r\left\{2r^2(R+B) - A^2\right\}}, \tag{11}$$

$$\frac{d^2}{d\phi^2}\left(\frac{u}{r}\right) + \frac{u}{r} = 0. \tag{12}$$

Integrating this we find

$$\left.\begin{aligned} u &= r\,(g_1 \cos\phi + h_1 \sin\phi) \\ \text{Similarly,}\quad v &= r\,(g_2 \cos\phi + h_2 \sin\phi) \\ x &= r\,(g_3 \cos\phi + h_3 \sin\phi) \end{aligned}\right\} \tag{13}$$

&c. &c.

But from (9) we have

$$t + a = \int \frac{r\,dr}{\left\{2r^2(R+B) - A^2\right\}^{\frac{1}{2}}}, \tag{14}$$

om (11)

$$\phi + \beta = \int \frac{A\,dr}{r\left\{2r^2(R+B) - A^2\right\}^{\frac{1}{2}}}. \tag{15}$$

By means of these we obtain ϕ as a function of r, and r as a function of $t + a$, and therefore ϕ as a function of $t + a$. Then the equations (13) will give u, v, x, &c. in terms of $t + a$, β, g_1, h_1, g_2, h_2, &c. A and B, the number of arbitrary constants being thus $2n + 4$. But there are relations subsisting between the constants which reduce the number of *independent* constants to $2n$. In the first place, the constant β will only alter g_1, h_1, g_2, h_2, &c., and it may therefore be neglected, so that the number of arbitrary constants is reduced to $2n + 3$. Again, since

$$r^2 = u^2 + v^2 + x^2 + \text{\&c.},$$

we have by squaring and adding equations (13)

$$1 = \cos^2\phi\, \Sigma(g^2) + \sin^2\phi\, \Sigma(h^2) + 2 \sin\phi \cos\phi\, \Sigma(gh).$$

In order that this equation may subsist for all values of ϕ we must have the conditions

$$\Sigma(g^2) = 1, \quad \Sigma(h^2) = 1, \quad \Sigma(gh) = 0.$$

These three conditions reduce the number of arbitrary constants to $2n$.

It is to be observed that the integrals for determining t and ϕ are not independent: for if we assume a function

$$S = \int \frac{dr}{r} \left\{2r^2 (R + B) - A^2\right\}^{\frac{1}{2}},$$

we have $$t + \alpha = \frac{dS}{dB}, \quad \phi + \beta = -\frac{dS}{dA}.$$

CHAPTER VIII.

SINGULAR SOLUTIONS OF DIFFERENTIAL EQUATIONS.

By a singular solution of a differential equation, is meant a certain relation between the variables which satisfies the differential equation, but does not satisfy the general integral. Solutions of this kind have long attracted the attention of mathematicians, and the memoirs in which they are discussed are very numerous. Their existence was first pointed out by Taylor, in his *Methodus Incrementorum*, p. 27, and afterwards they were noticed by Clairaut, in the *Mémoires de l'Académie des Sciences* for 1734. But Euler, in the *Mémoires de l'Académie de Berlin* for 1756, was the first who considered the subject in its bearing on the general Theory of Integration; and in his Integral Calculus, Vol. I. Sect. 2, Chap. IV., he gave a test for discovering whether a given solution be or be not included in the general integral. Lagrange, in the *Mémoires de l'Académie de Berlin*, and afterwards in his *Théorie des Fonctions*, and his *Calcul des Fonctions*, discussed the theory of these solutions, and shewed the connection between them and the general integral, and their relative geometric interpretations. Other points of the theory have been elucidated by Laplace (*Mémoires de l'Académie des Sciences*, 1772), Legendre (*Ib.* 1790), and Poisson, *Journal de l'Ecole Polytechnique*, Cahier XIII.

Having given a differential equation, to find its singular solutions if it have any.

Let $$U = 0$$

be a differential equation of the first order between x and y cleared of radicals and fractions, then if we represent $\frac{dy}{dx}$ by p, the relations between x and y found by eliminating p between

$$U = 0, \quad \text{and} \quad \frac{dU}{dp} = 0,$$

are singular solutions of $U = 0$, provided they satisfy that equation, and do not at the same time make $\frac{dU}{dy} = 0$. We might also deduce the singular solutions from eliminating $\frac{dx}{dy}$ between

$$U = 0, \quad \text{and} \quad \frac{dU}{dp_1} = 0,$$

where $p_1 = \frac{dx}{dy}$, provided that they do not at the same time make $\frac{dU}{dx} = 0$.*

Ex. (1) Let the equation be

$$x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} + m = 0.$$

Here
$$\frac{dU}{dp} = 2xp - y = 0;$$

and eliminating p between this and the preceding equation, we find

$$y^2 - 4mx = 0,$$

as the singular solution.

(2) Let $y + (y - x)\frac{dy}{dx} + (a - x)\left(\frac{dy}{dx}\right)^2 = 0$, be the given equation. Then

$$(x + y)^2 - 4ay = 0$$

is the singular solution.

(3) Let $y^2 - 2xy\frac{dy}{dx} + (1 + x^2)\left(\frac{dy}{dx}\right)^2 = 1$ be the given equation: the singular solution is

$$y^2 = 1 + x^2.$$

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* Laplace, *Mémoires de l'Académie*, 1772.

This is the equation with respect to which Taylor first made the remark that it admitted of a solution not involved in the general integral—"singularis quædam solutio," as he terms it. See his *Methodus Incrementorum*, p. 27.

(4) Let $x^2 + 2xy\frac{dy}{dx} + (a^2 - x^2)\left(\frac{dy}{dx}\right)^2 = 0$.

The singular solution of this is

$$x^2 + y^2 - a^2 = 0.$$

(5) Let $\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} + x = 0$.

The equation resulting from the elimination of $\frac{dy}{dx}$ between this equation and

$$2\frac{dy}{dx} + y = 0,$$

is

$$y^2 - 4x = 0;$$

but as this does not satisfy the given equation it is not a singular solution.

(6) Let $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\left(y - x\frac{dy}{dx}\right)^2 = a^2\left(\frac{dy}{dx}\right)^2$.

In this case $\frac{dU}{dp} = 0$ gives us

$$x = \frac{a}{(1 + p^2)^{\frac{3}{2}}}.$$

Eliminating p by means of this equation we find as the singular solution

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

(7) Let $y^2\left(\frac{dy}{dx}\right)^2 - 2xy\left(\frac{dy}{dx}\right) + ax + by = 0$.

Here $\frac{dU}{dp} = 0$ gives us $p = \frac{x}{y}$; and the result of the elimination of p is

$$ax + by - x^2 = 0;$$

but as this does not satisfy the given equation it is no solution at all.

(8) Let the equation be

$$(x\frac{dy}{dx}-y)(x\frac{dy}{dx}-2y)+x^3=0.$$

The singular solution is

$$y^2-4x^3=0.$$

When a solution of an equation is given, and it is required to find whether it be a singular solution or a particular integral, we must deduce from it the value of $p=\frac{dy}{dx}$, and see whether when substituted in $\frac{dU}{dp}$ it make it vanish, and do not at the same time make $\frac{dU}{dy}$ vanish. If this be the case it is a singular solution, otherwise it is a particular integral.

(9) Are $y^2=2x+1$, and $y^2+x^2=0$, singular solutions or particular integrals of the equation

$$y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0?$$

Here $$\frac{dU}{dp}=2(yp+x).$$

Now from the first of the given solutions we find

$$p=\frac{1}{y},$$

which does not make $\frac{dU}{dp}$ vanish: it is therefore a particular integral.

From the second of the given solutions we find

$$p=-\frac{x}{y},$$

which does make $\frac{dU}{dp} = 0$; and as it does not make $\frac{dU}{dy}$ vanish, it is a singular solution.

(10) In the same way it will be seen that

$$y^2 + (x-1)^2 = 0,$$

is a particular integral of

$$\left(\frac{dy}{dx}\right)^2 + y\frac{dy}{dx} + x = 0.$$

If the differential equation be of an order higher than the first, let $y_1, y_2 \dots y_n$ represent the successive differential coefficients of y with respect to x. Then

$$U = 0$$

being the equation cleared of fractions and radicals as before, the conditions that $y_{n-m} = X$ should be a singular solution of the $(n-m)^{\text{th}}$ order are

$$\frac{dU}{dy_n} = 0,\quad \frac{dU}{dy_{n-1}} = 0 \dots \frac{dU}{dy_{n-m+1}} = 0;$$

and therefore if we find a relation between x, y, and $y_{n-m} = X$, which satisfies these equations and also the given equation, it is a singular solution of the $(n-m)^{\text{th}}$ order.

Legendre, *Mém. de l'Acad.*, 1790, p. 218.

(11) Let $x\left(\frac{d^2y}{dx^2}\right)^2 - 2\frac{dy}{dx}\frac{d^2y}{dx^2} + x = 0$

be the given equation. Putting $\frac{dy}{dx} = y_1$, $\frac{d^2y}{dx^2} = y_2$ we may write it

$$xy_2^2 - 2y_1y_2 + x = 0.$$

Here $$\frac{dU}{dy_2} = 2(xy_2 - y_1) = 0$$

gives $y_2 = \frac{y_1}{x}$; and, by means of this, eliminating y_2 from the original equation we find

$$y_1^2 - x^2 = 0$$

as a singular solution of the first order. As this does not satisfy $\dfrac{dU}{dy_1} = 0$, and as there is only one factor in $\dfrac{dU}{dy_2} = 0$, there is no singular solution of the final integral.

(12) Let

$$y - x\frac{dy}{dx} + \frac{x^2}{2}\frac{d^2y}{dx^2} - \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)^2 - \left(\frac{d^2y}{dx^2}\right)^2 = 0.$$

The condition $\dfrac{dU}{dy_2} = 0$ gives us

$$y_2 = \frac{4xy_1 + x^2}{4(1 + x^2)},$$

from which we find the singular solution of the first order to be

$$\left(\frac{dy}{dx}\right)^2 + x\left(1 + \frac{x^2}{2}\right)\frac{dy}{dx} - y(1 + x^2) - \frac{x^4}{16} = 0.$$

(13) Let the equation be

$$(xy - 1)\left\{2xy\frac{d^2y}{dx^2} - y\frac{dy}{dx} + x\left(\frac{dy}{dx}\right)^2\right\}^2 - xy\left(y + x\frac{dy}{dx}\right)^2\left(\frac{dy}{dx}\right)^2 = 0.$$

It will be found that

$$xy - 1 = 0$$

satisfies $\dfrac{dU}{dy_2} = 0$, $\dfrac{dU}{dy_1} = 0$ and $U = 0$, and as it is independent of y_1 and y_2 it is the singular solution belonging to the final integral.

(14) If the equation be of the third order

$$\left(1 + \frac{d^3y}{dx^3}\right)^2 - 4\left(\frac{dy}{dx}\right)^2\left(x + \frac{d^2y}{dx^2}\right) = 0,$$

a singular solution of the second order is

$$\frac{d^2y}{dx^2} + x = 0.$$

(15) Let the equation be

$$a^2\left(1 + \frac{d^3y}{dx^3}\right)^2 - 4\left(\frac{dy}{dx}\right)^2\left(x + \frac{d^2y}{dx^2}\right)^3 = 0.$$

Then $y_2 + x = 0$ is not a singular solution, because though it make $\dfrac{dU}{dy_3} = 0$ it also causes $\dfrac{dU}{dy_2}$ to vanish; and we find that the real value of $\dfrac{dy_2}{dy_3}$ is infinity instead of zero, as it would be in the case of a singular solution.

Having given the general integral of a differential equation of the first order, to find the singular solutions of the equation when there are such.

Let $u = 0$ be the integral cleared of radicals. As it is supposed to be an integral of an equation of the first order, it must contain an arbitrary constant, which we shall call c. Then if the equation

$$\frac{dU}{dc} = 0$$

give a value of c in terms of x and y, the elimination of c between $U = 0$ and $\dfrac{dU}{dc} = 0$ will give an equation in x and y, which is the singular solution. It is to be observed that if $\dfrac{dU}{dc} = 0$ give a constant value for c, or a value in terms of x and y, which becomes constant in consequence of the relation $U = 0$, the result of the elimination is not a singular solution but a particular integral.

(16) Let the equation be

$$x^2 - 2cy - c^2 - a^2 = 0.$$

Then $$\frac{dU}{dc} = -2(y + c) = 0,$$

whence $c = -y$, so that $x^2 + y^2 - a^2 = 0$

is the required singular solution.

(17) Let $$y - ax - \frac{x^2}{2}(c - a)^2 + \frac{b^2}{4}(c - a)^4 = 0.$$

Then $$\frac{dU}{dc} = b^2(c - a)^3 - x^2(c - a) = 0.$$

This is satified either by $c = a$ or $c = a \pm \frac{x}{b}$.

The former gives a particular integral. The latter gives

$$4b^2 (y - ax) - x^4 = 0,$$

which is the singular solution.

(18) Let $(x^2 + y^2 - a^2)(y^2 - 2cy) + (x^2 - a^2) c^2 = 0$.

Then $$\frac{dU}{dc} = 2 \{c (x^2 - a^2) - y (x^2 + y^2 - a^2)\} = 0;$$

from which $$c = \frac{y (x^2 + y^2 - a^2)}{x^2 - a^2},$$

which being substituted in the equation gives

$$x^2 + y^2 - a^2 = 0$$

as the singular solution: but since this makes $c = 0$, it appears that it is only a particular integral found by making the arbitrary constant equal to zero.

Let $U = 0$ be the integral of an equation of the second order, so that it contains two arbitrary constants c_1, c_2; then, if we represent the differentiation with respect to x and y by d, and that with respect to c_1 and c_2 by d', we can obtain the singular solution by eliminating c_1, c_2, and $\frac{dc_2}{dc_1}$ between the equations

$$U = 0, \quad dU = 0, \quad d'U = 0, \quad dd'U = 0.$$

(19) Let the given integral be

$$y = \tfrac{1}{2} c_1 x^2 + c_2 x + c_1^2 + c_2^2,$$

so that $$U = \tfrac{1}{2} c_1 x^2 + c_2 x + c_1^2 + c_2^2 - y = 0.$$

Then $$dU = (c_1 x + c_2)\, dx - dy = 0,$$

$$d'U = (\tfrac{1}{2} x^2 + 2c_1)\, dc_1 + (x + 2c_2)\, dc_2 = 0,$$

$$dd'U = (x\, dc_1 + dc_2)\, dx = 0.$$

From the last we find

$$\frac{dc_2}{dc_1} = -x.$$

Substituting this in the preceding equation we have

$$4(c_1 - c_2 x) - x^2 = 0.$$

Between this equation, and the first two we can eliminate c_1 and c_2, and we find as the singular solution of the first integral

$$\left(\frac{dy}{dx}\right)^2 + (\tfrac{1}{2}x^2 + 1)\,x\,\frac{dy}{dx} - \frac{x^4}{16} - y(1 + x^2) = 0.$$

(20) Let the integral be

$$y = \tfrac{1}{2}c_1 x^2 + c_2 x + c_1 c_2.$$

By a similar process to that in the last example, we find as the singular solution belonging to the first integral of the differential equation

$$\frac{x^2}{4} + \left(\frac{dy}{dx}\right)^2 + 3x^2\frac{dy}{dx} - 4xy = 0.$$

There is no singular solution belonging to the final integral, but the singular solution just found has itself a singular solution, which is

$$x^3 + 2y = 0.$$

Singular Solutions of Partial Differential Equations.

If $U = 0$ be a partial differential equation of the first order in x, y, and z, and if we put $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, the singular solution, if there be one, will be found by eliminating p and q between the three equations

$$U = 0, \quad \frac{dU}{dp} = 0, \quad \frac{dU}{dq} = 0.$$

(21) Let the equation be

$$(z - px - qy)^2 = a^2(1 + p^2 + q^2).$$

Then $$\frac{dU}{dp} = -(z - px - qy)\,x - a^2 p = 0,$$

$$\frac{dU}{dq} = -(z - px - qy)\,y - a^2 q = 0.$$

By means of these eliminating p and q, we find

$$x^2 + y^2 + z^2 = a^2$$

as the singular solution.

(22) Let the equation be

$$(px - qy)^2 q + 4mx^3 (z - px) = 0.$$

$$\frac{dU}{dp} = q(px - qy) - 2mx^3 = 0,$$

$$\frac{dU}{dq} = (px - qy)(px - 3qy) = 0.$$

These two equations agree with the original one, if we assume

$$px - 3qy = 0;$$

and the singular solution found by eliminating p and q is

$$z^2 - 4mx^3 y = 0.$$

Legendre, *Mémoires de l' Académie*, 1790, p. 238.

(23) Let $(z - px - qy)^m = Ap^a q^b$.

$$\frac{dU}{dp} = -mx(z - px - qy)^{m-1} - aAp^{a-1}q^b = 0,$$

$$\frac{dU}{dq} = -my(z - px - qy)^{m-1} - bAp^a q^{b-1} = 0.$$

Dividing the first of these by the second we find

$$\frac{aq}{bp} = \frac{x}{y}.$$

Dividing the original equation by the first we have

$$\frac{z - px - qy}{mx} = -\frac{p}{a}.$$

Eliminating q between these we find

$$p = -\frac{az}{cx}, \text{ where } c = m - (a + b).$$

Similarly $q = -\dfrac{bz}{cy}$.

By means of these values, eliminating p and q, we find the singular solution to be

$$x^a y^b z^c = (-)^{a+b} A \,.\, \frac{a^a b^b c^c}{m^m}.$$

If the partial differential equation be of the second order and we put

$$\frac{d^2 z}{dx^2} = r, \quad \frac{d^2 z}{dx\, dy} = s, \quad \frac{d^2 z}{dy^2} = t,$$

the conditions which must be satisfied in order that an equation should be the singular solution of the first order of the equation $U = 0$ are

$$\frac{dU}{dr} = 0, \quad \frac{dU}{ds} = 0, \quad \frac{dU}{dt} = 0.$$

If the function is to be a singular solution belonging to the final integral, it must in addition satisfy the equations

$$\frac{dU}{dp} = 0, \quad \frac{dU}{dq} = 0.$$

(24) Let the given equation be

$$r^2 - 2qr\left(p - \frac{z}{1+x}\right) + \left(p - \frac{z}{1+x}\right) y = 0.$$

Here $\frac{dU}{dr} = 0$ is the only condition, and we have

$$\frac{dU}{dr} = 2\left\{r - q\left(p - \frac{z}{1+x}\right)\right\} = 0.$$

Comparing this with the given equation, we find

$$p - \frac{z}{1+x} = 0,$$

from which $$r = \frac{1}{1+x}\left(p - \frac{z}{1+x}\right) = 0,$$

which satisfies both $U = 0$ and $\frac{dU}{dr} = 0$: therefore

$$p - \frac{z}{1 + x} = 0$$

is the singular solution required. The integral of this is

$$z = (1 + x)\,\phi(y).$$

Poisson, *Jour. de l'Ecole Polyt.* Cah. XIII. p. 113.

(25) Let the equation be

$$r^2 - t^2 + \left(\frac{p}{x} - \frac{q}{y}\right)(z - x - y) = 0.$$

It will be found that

$$z = x + y$$

satisfies the original equation as well as

$$\frac{dU}{dr} = 0, \quad \frac{dU}{dt} = 0, \quad \frac{dU}{dp} = 0, \quad \frac{dU}{dq} = 0;$$

it is therefore a singular solution corresponding to the final integral.

CHAPTER IX.

QUADRATURE OF AREAS AND SURFACES, RECTIFICATION OF CURVES AND CUBATURE OF SOLIDS.

SECT. 1. *Quadrature of Plane Areas.*

WHEN an area is referred to rectangular co-ordinates x and y, the double integral $\iint dx\,dy$ taken between the proper limits gives the value of the area. One of the integrations may always be performed, so that we have either

$$\int y\,dx + C \quad \text{or} \quad \int x\,dy + C,$$

and these integrals are to be taken between the limits of y or x, which form the boundaries of the area. If we take the first of these expressions, the limiting values of y must either be constants or functions of x given by the equation to the bounding curve: therefore on substituting these values we obtain a function of x alone, which is to be integrated, and taken between the limits of that variable which are required by the problem. If after the first integration we suppose $C = 0$, the integral $A = \int y\,dx$ expresses the area included between the axis of x, the curve, and two ordinates corresponding to the limits of x.

In taking the integral $\int y\,dx$ between the final limits of x, it is necessary that the interval should not contain a value of x which causes y to vanish or become infinite, as in that case we might be led to an erroneous conclusion. Thus if we suppose a curve to be symmetrically situate in the first and third quadrants, and to intersect the axis at the origin; and if we were to integrate from $x = a$ to $x = -a$ we should obtain zero as our result, instead of finding the area to be double of that from $x = 0$ to $x = a$, or that from $x = 0$ to $x = -a$. Therefore when any interval from a to b contains a value c of x which makes y vanish or become infinite, we must break it up into two intervals, one from b to c and the other from c to a, and add the integrals corresponding to these. In like manner if the interval contain several values of x which make y vanish or become infinite, we must split it up into as many smaller intervals, each having one of these values of x as a limit, and add them all together.

If the co-ordinates be not rectangular, and α be the angle between them, we must multiply the integral by $\sin\alpha$ to obtain the value of the area.

Ex. (1) If we take the general equation to a parabola of any order

$$y^{m+n} = a^m x^n,$$

we have $$A = \int y\,dx = \frac{m+n}{m+2n}\, a^{\frac{m}{m+n}}\, x^{\frac{m+2n}{m+n}} + C.$$

(2) The general equation to hyperbolas referred to their asymptotes is

$$x^m y^n = a^{m+n},$$

$$A = \int y\,dx = C + \frac{n}{n-m}\, a^{\frac{m+n}{n}}\, x^{\frac{n-m}{n}}.$$

This formula fails when $m = n$, in which case

$$A = C + a^2 \log x.$$

(3) When the common hyperbola is referred to its axes, the sectorial area ACP (fig. 53) is easily found.

For $$ACP = NCP - ANP = \tfrac{1}{2}xy - \int y\,dx.$$

Now $$y = \frac{b}{a}(x^2 - a^2)^{\frac{1}{2}},$$

and $$\int y\,dx = \frac{b}{2a}\left\{x(x^2-a^2)^{\frac{1}{2}}\right\} - a^2 \log\left\{x + (x^2-a^2)^{\frac{1}{2}}\right\} + C.$$

Determining the constant by the condition that the area vanishes when $x = a$, we have

$$\int y\,dx = \tfrac{1}{2}xy - \tfrac{1}{2}ab\,\frac{\log\left\{x + (x^2-a^2)^{\frac{1}{2}}\right\}}{a},$$

so that $$ACP = \frac{ab}{2}\log\left(\frac{x}{a} + \frac{y}{b}\right).$$

(4) In the circle, the equation to which is

$$x^2+y^2=a^2,$$

$$A=\int dx\,(a^2-x^2)^{\frac{1}{2}}=\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C.$$

If this be taken from $x=a$ to $x=0$, we find the area of the quadrant to be $\frac{\pi a^2}{4}$, and therefore that of the whole circle to be πa^2.

If the equation to the circle be

$$y^2=2ax-x^2,$$

$$A=\int dx\,(2ax-x^2)^{\frac{1}{2}}=\frac{x-a}{2}(2ax-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\text{vers}^{-1}\frac{x}{a}.$$

(5) The equation to the ellipse being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$A=\frac{b}{a}\int dx\,(a^2-x^2)^{\frac{1}{2}}=\frac{b}{a}\,.\,\text{circ. area whose cosine is }\frac{x}{a}+C.$$

For the whole ellipse $A=\frac{b}{a}\pi a^2=\pi ab$.

(6) The equation to the witch of Agnesi is

$$xy^2=4a^2(2a-x).$$

$$A=2a\left\{(2ax-x^2)^{\frac{1}{2}}+a\,\text{vers}^{-1}\frac{x}{a}\right\}+C.$$

Taking this from $x=2a$ to $x=0$, and doubling it on account of the symmetry on both sides of the axis of x, we find the whole area between the curve and its asymptote to be $4\pi a^2$.

(7) The equation to the cissoid is

$$y^2(2a-x)=x^3.$$

Here

$$A=-2x(2ax-x^2)^{\frac{1}{2}}+3\,.\,\text{circ. area whose versine is }\frac{x}{a}+C,$$

and the whole area included between the asymptote and the two branches of the curve is $3\pi a^2$.

(8) The equation to the cycloid is

$$\frac{dy}{dx} = \frac{(2ax - x^2)^{\frac{1}{2}}}{x}.$$

This being only a differential equation, it is necessary to use an artifice for the purpose of effecting the integration. If we integrate $\int y\,dx$ by parts we have

$$\begin{aligned}\int y\,dx &= xy - \int x\,dy,\\ &= xy - \int dx\,(2ax - x^2)^{\frac{1}{2}}.\end{aligned}$$

Taking this integral from $x = 0$ to $x = 2a$, and doubling it, we find the whole area of the cycloid to be $3\pi a^2$, or three times the area of the generating circle.

(9) The differential equation to the tractrix being

$$\frac{dy}{dx} = -\frac{y}{(a^2 - y^2)^{\frac{1}{2}}},$$

$$A = \int y\,dx = -\int dy\,(a^2 - y^2)^{\frac{1}{2}};$$

and the whole area included between the curve and the positive axes is $\dfrac{\pi a^2}{4}$.

(10) The equation to the catenary being

$$y = \tfrac{1}{2}c\,(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}),$$

its area is $\qquad \tfrac{1}{2}c^2\,(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}) = c\,(y^2 - c^2)^{\frac{1}{2}}.$

(11) The equation to the evolute of the ellipse is

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{\beta}\right)^{\frac{2}{3}} = 1.$$

The whole area inclosed by the curve is $\dfrac{3\pi a\beta}{8}$.

This is best investigated by Divichlet's method of evaluating definite integrals. See Chap. XI.

(12) The equations to the companion to the cycloid are

$$y = a\theta, \quad x = a(1 - \cos\theta),$$

$$\int y\,dx = xy - \int x\,dy = a^2(\sin\theta - \theta\cos\theta) + C.$$

The whole area is $2\pi a^2$, or twice the area of the generating circle.

When an area is referred to polar co-ordinates r and θ, its value is given by the double integral $\int\int r\,dr\,d\theta$ taken between proper limits. Integrating with respect to r we have $A = \frac{1}{2}\int r^2\,d\theta + C$; and if we suppose $C = 0$, the integral $A = \frac{1}{2}\int r^2\,d\theta$, in which there is substituted for r its value in terms of θ given by the equation to the curve, is the value of the sectorial area swept out by the radius vector. In taking the integral between the limiting values of θ, the same precaution must be observed as in the case of rectilinear co-ordinates, that the interval shall not contain a value of θ which causes r to vanish or become infinite. If we suppose θ to increase indefinitely, the same geometrical space will be repeatedly swept over by the radius vector at each revolution, so that, when the curve is not re-entering, the analytical area (if we may use the phrase) differs from the geometrical area: to obtain the latter we must subtract from the analytical area that portion which has been previously swept over. Thus if we wish to find the geometrical area included between the values 0 and 4π of θ, and if we put

$$A_1 = \tfrac{1}{2}\int_0^{2\pi} r^2 d\theta, \quad A_2 = \tfrac{1}{2}\int_0^{4\pi} r^2 d\theta,$$

the required area is $A_2 - A_1$.

(13) The equation to the Lemniscate is

$$r^2 = a^2\cos 2\theta,$$

$$A = \tfrac{1}{2}\int r^2 d\theta = \tfrac{1}{2}a^2\int d\theta\cos 2\theta = C + \tfrac{1}{4}a^2\sin 2\theta.$$

If we take this from $\theta = 0$ to $\theta = \frac{1}{4}\pi$, we have

$$A = \tfrac{1}{4}a^2.$$

This is the fourth part of the whole area of the curve, which is therefore equal to a^2.

In this case, if we had at once integrated from $\theta = 0$ to $\theta = \pi$, or $\theta = 2\pi$ we should have found the area to be zero.

This anomaly would arise from our integrating through an interval in which r becomes zero.

(14) Let the equation to the curve be

$$r = a\cos\theta + b, \text{ where } a > b.$$

The form of this curve is given in fig. 42.

If we wish to find the area included within *ODCAHG*, it is sufficient to integrate from $\theta = 0$ to that value of θ which causes r to vanish, and then to double the result. Let $\alpha = \cos^{-1}\left(-\frac{b}{a}\right)$, then the area *ODCAHG* is equal to

$$\tfrac{1}{2}\left\{(a^2 + 2b^2)\,\alpha + 3b\,(a^2 - b^2)^{\frac{1}{2}}\right\};$$

and the area *OEBF* is equal to

$$\tfrac{1}{2}\left\{(a^2 + 2b^2)\,(\pi - \alpha) - 3b\,(a^2 - b^2)^{\frac{1}{2}}\right\}.$$

If $b = a$, the curve becomes the common cardioid, and its area is $\frac{3\pi a^2}{2}$.

(15) The equation to the conchoid of Nicomedes when referred to polar co-ordinates is

$$r = a\sec\theta + b,$$

and its area is

$$\tfrac{1}{2}\left\{a^2\tan\theta + 2ab\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + b^2\theta\right\} + C.$$

(16) The curve whose equation is

$$r = a\sin 3\theta$$

has six loops (see fig. 49), and it is sufficient to find the area inclosed by one of them. This is easily seen to be $\frac{\pi a^2}{12}$ and therefore the sum of the areas of the six loops is $\frac{1}{2}\pi a^2$, or one half of the area of the circle which bounds them.

(17) The equation to the spiral of Archimedes is

$$r = a\theta.$$

Hence the area $= \frac{a^2\theta^3}{6} + C.$

After n revolutions the analytical area swept out is $a^2\dfrac{n^3(2\pi)^3}{6}$; but to obtain the geometrical area we must subtract from it the area corresponding to $(n-1)$ revolutions, which gives us $(3n^2-3n+1)\dfrac{(2\pi)^3a^2}{6}$ as the required geometrical area. In the same way we should obtain as the geometrical area corresponding to $(n+1)$ revolutions, the expression $(3n^2+3n+1)(2\pi)^3\dfrac{a^2}{6}$, and the difference between these or the space between the arcs after $(n+1)$ and after n revolutions is $n(2\pi)^3a^2$, which is n times the space between the arcs after the first and second revolutions.

(18) In the hyperbolic spiral

$$r\theta = a.$$

The area swept out by the radius vector from 0 to r is $\frac{1}{2}ar$, which is equal to the triangle formed by the radius, the tangent and the sub-tangent.

If the equation to the spiral be given by a relation between p and r, we have

$$A = \tfrac{1}{2}\int\frac{prdr}{(r^2-p^2)^{\frac{1}{2}}}.$$

(19) In the involute of the circle

$$r^2 - p^2 = a^2.$$

Therefore $A = \dfrac{1}{2a}\int dr\, r\,(r^2-a^2)^{\frac{1}{2}} = \dfrac{p^3}{6a} + C.$

(20) In the epicycloid

$$p^2 = \frac{c^2(r^2-a^2)}{c^2-a^2},$$

where $c = a + 2b$, a and b being the radii of the fixed and generating circles respectively. Hence

$$A = \tfrac{1}{2}\frac{c}{a}\int r\,dr\left(\frac{r^2-a^2}{c^2-r^2}\right)^{\frac{1}{2}} = \tfrac{1}{2}\frac{c}{a}\int\frac{r\,dr\,(r^2-a^2)^{\frac{1}{2}}}{\{c^2-a^2-(r^2-a^2)\}^{\frac{1}{2}}}$$

$$= \frac{-c(r^2-a^2)^{\frac{1}{2}}(c^2-r^2)^{\frac{1}{2}}}{4a} + \frac{c(c^2-a^2)}{4a}\sin^{-1}\left(\frac{r^2-a^2}{c^2-a^2}\right)^{\frac{1}{2}} + C.$$

Hence the area swept out by r during one revolution of the generating circle is

$$\frac{c\,(c^2-a^2)\,\pi}{4a}=\frac{\pi b}{a}\,(a^2+3ab+2b^2).$$

Subtracting from this the area of the sector of the fixed circle which is πab, we have for the area included between the epicycloid and the fixed circle

$$A=\frac{\pi b^2}{a}\,(3a+2b).$$

When a curve forms a loop, the area may sometimes be conveniently found by taking $\frac{y}{x}$, or the tangent of the angle which the radius makes with the axis of x, as the independent variable. If we put $\frac{y}{x}=\tan\theta=t$, we have $d\theta=dt\cos^2\theta$ and

$$A=\tfrac{1}{2}\int r^2d\theta=\tfrac{1}{2}\int d\theta x^2\sec^2\theta=\tfrac{1}{2}\int dt\,x^2.$$

(21) The curve $y^3-3axy+x^3=0$,

has a loop which touches the axes of x and y at the origin; see fig. 51. Now putting $y=xt$, we find

$$x=\frac{3at}{1+t^3},$$

and $$A=\frac{9a^2}{2}\int dt\frac{t^2}{(1+t^3)^2}=-\frac{3a^2}{2}\,\frac{1}{1+t^3}+C;$$

and taking this from $t=0$ to $t=\infty$ we have

$$A=\frac{3a^2}{2}\text{ for the whole area of the loop.}$$

(22) The lemniscate whose equation is

$$(x^2+y^2)^2=a^2x^2-b^2y^2,$$

has two loops; find its area.

$$A=\tfrac{1}{2}\int dt\,x^2=\tfrac{1}{2}\int dt\,\frac{a^2-b^2t^2}{(1+t^2)^2},$$

and for each loop the limits of t are $\frac{a}{b}$ and $-\frac{a}{b}$. The whole area is $ab + (a^2 - b^2)\tan^{-1}\frac{a}{b}$.

SECT. 2. *Rectification of Curves.*

When a curve is referred to rectangular co-ordinates, the length of any portion of it is found by integrating

$$\int dx \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$$

between the proper limits.

(1) The equation to the common parabola being

$$y^2 = 4mx,$$

the length of an arc measured from the vertex is

$$\int dx \left(\frac{x+m}{x}\right)^{\frac{1}{2}} = (x^2 + mx)^{\frac{1}{2}} + \frac{m}{2}\log\frac{m + 2\{x + (x^2 + mx)^{\frac{1}{2}}\}}{m}.$$

(2) The equation to the semicubical parabola is

$$ay^2 = x^3.$$

The length of the arc measured from the origin is

$$s = \frac{(4a + 9x)^{\frac{3}{2}} - (4a)^{\frac{3}{2}}}{27a^{\frac{1}{2}}}.$$

This was the first curve which was rectified. The author was William Neil, who was led to the discovery by a remark of Wallis in his Arithmetica Infinitorum. See *Wallisii Opera*, Tom. I. p. 551.

(3) To find when the curves expressed by the equation

$$a^m y^n = x^{m+n}$$

are rectifiable. We have

$$s = \int dx \left\{1 + \left(\frac{m+n}{n}\right)^2 a^{-\frac{2m}{n}} x^{\frac{2m}{n}}\right\}^{\frac{1}{2}}.$$

This is integrable when $\frac{n}{2m}$ or $\frac{n}{2m} + \frac{1}{2}$ is an integer: the first of these gives

$$\frac{m+n}{n} = \frac{3}{2} = \frac{5}{4} = \frac{7}{6} = \frac{9}{8} \ldots\ldots,$$

the second gives

$$\frac{m+n}{n} = \frac{2}{1} = \frac{4}{3} = \frac{6}{5} = \text{\&c.}$$

(4) The equation to the cycloid being

$$\frac{dy}{dx} = \frac{(2ax - x^2)^{\frac{1}{2}}}{x},$$

we have $$s = 2(2ax)^{\frac{1}{2}} + C.$$

Hence the whole length of the cycloid is $8a$ or four times the diameter of the generating circle. This rectification was discovered by Wren.

(5) The equation to one of the hypocycloids is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}};$$

the whole length of the curve is $6a$.

(6) The equation to the catenary is

$$y = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} + \epsilon^{-\frac{x}{c}}\right).$$

Then $$\frac{ds}{dy} = \frac{y}{(y^2 - c^2)^{\frac{1}{2}}},$$

and $$s = (y^2 - c^2)^{\frac{1}{2}} = \frac{c}{2}\left(\epsilon^{\frac{x}{c}} - \epsilon^{-\frac{x}{c}}\right),$$

the arc being measured from the point where $y = c$.

Hence as the area is equal to $c(y^2 - c^2)^{\frac{1}{2}}$, it is equal to cs; that is, the area contained between the axes, the curve and any ordinate is equal to the length of the corresponding arc multiplied by a constant.

(7) The equation to the tractory is

$$\frac{dy}{dx} + \frac{y}{(a^2 - y^2)^{\frac{1}{2}}} = 0.$$

Then $$s = a \log \frac{y}{a},$$

supposing the arc to be measured from the point where $y = a$.

When a curve is the evolute of another curve, the length of its arc is best found by taking the difference of the radii of curvature of the involute, corresponding to the extremities of the arc.

(8) To find the length of the evolute of the ellipse.

The radius of curvature of the ellipse at the extremity of the major-axis is $\frac{b^2}{a}$; that at the extremity of the minor-axis is $\frac{a^2}{b}$: therefore the length of the fourth part of the evolute is $\frac{a^2}{b} - \frac{b^2}{a} = \frac{a^3 - b^3}{ab}$.

If the curve be referred to polar co-ordinates r and θ,

$$s = \int d\theta \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}};$$

and if it be referred to p and r

$$s = \int dr \frac{r}{(r^2 - p^2)^{\frac{1}{2}}}.$$

(9) In the logarithmic spiral

$$r = c\epsilon^{a\theta}.$$

Therefore $s = \int d\theta \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} = (1 + a^2)^{\frac{1}{2}} \frac{r}{a}$,

supposing it to be measured from the pole. Hence the arc is equal to the portion of the tangent at its extremity, which is intercepted between the point of contact and the subtangent.

(10) The equation to the spiral of Archimedes being

$$r = a\theta,$$

the length of the arc from the origin is

$$\frac{r(a^2 + r^2)^{\frac{1}{2}}}{2a} + \frac{a}{2} \log \frac{r + (a^2 + r^2)^{\frac{1}{2}}}{2a}$$

This is the same as the arc of a parabola (whose latus rectum is $2a$) intercepted between the vertex and an ordinate equal to r.

(11) In the involute of the circle the equation to which is

$$r^2 - p^2 = a^2,$$

$$s = \int \frac{r\,dr}{a} = \frac{r^2}{2a} + C.$$

If the arc be measured from the point where $r = a$ we find $c = -\frac{a}{2}$, and $s = \frac{p^2}{2a}$. Now p is always equal to the length of the arc of the circle which is unwound, so that if this be called $a\theta$,

$$s = \tfrac{1}{2}a\theta^2.$$

(12) The equation to the epicycloid is

$$p^2 = \frac{c^2(r^2 - a^2)}{c^2 - a^2},$$

where $c = a + 2b$, a and b being the radii of the fixed and generating circles respectively. Hence

$$s = \frac{(c^2 - a^2)^{\frac{1}{2}}}{a} \int \frac{r\,dr}{(c^2 - r^2)^{\frac{1}{2}}} = -\frac{(c^2 - a^2)}{a}(c^2 - r^2)^{\frac{1}{2}}.$$

The whole arc corresponding to one revolution of the generating circle is $8\frac{b}{a}(a + b)$.

The corresponding arc in the hypocycloid is $8\frac{b}{a}(a - b)$.

For a curve of double curvature we have

$$s = \int dx \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}.$$

(13) In the helix,

$$y = a\cos nx, \quad z = a\sin nx;$$

therefore $s = \int dx\,(1 + n^2a^2)^{\frac{1}{2}} = (1 + n^2a^2)^{\frac{1}{2}}x + C$.

If the arc be measured from the origin $C = 0$, and

$$s = (1 + n^2a^2)^{\frac{1}{2}}x.$$

If θ be the constant angle which the curve makes with the axis of x, $na = \tan\theta$ and $(1 + n^2a^2)^{\frac{1}{2}} = \sec\theta$ so that $s = x \sec\theta$.

(14) The loxodrome is defined by the two equations

$$(x^2 + y^2)^{\frac{1}{2}} \left(\epsilon^{n\tan^{-1}\frac{y}{x}} + \epsilon^{-n\tan^{-1}\frac{y}{x}}\right) = 2r,$$

$$x^2 + y^2 + z^2 = r^2.$$

Changing into polar co-ordinates by the formulæ

$$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi,$$

we find
$$\frac{ds}{d\phi} = \frac{(1 + n^2)^{\frac{1}{2}}}{n};$$

and integrating from $\phi = 0$ to $\phi = \pi$ we have

$$s = \pi\frac{(1 + n^2)^{\frac{1}{2}}}{n}.$$

SECT. 3. *Cubature of Solids.*

If a solid be referred to rectangular co-ordinates its volume (V) is found by integrating the triple integral $\iiint dx\,dy\,dz$. If we integrate first with respect to z, and suppose the integral to begin when $z = 0$,

$$V = \iint z\,dx\,dy$$

is the volume, z being given in terms of x and y by the equation to the surface, and the integrals being taken between the proper limits, which differ according to the nature of the surfaces which bound the surface laterally. The most simple case is when the solid is bounded laterally by four planes, two parallel to the plane of xz, and two to that of yz. The limits of x and y being then constant are independent of each other, and the integration may be easily effected. But if the surface be terminated laterally by the curve surface, the extreme values of the variables are connected together by a relation derived from the equation to the surface by making $z = 0$. If we integrate first with respect to y, and if the equation to the surface give us, on making $z = 0$, a relation $f(x, y) = 0$ between x and y, we have to take the values of y in terms of x derived from this equation as the limits of the integral with respect to y. There remains now only to integrate a function of x, and to take it between the limits of the value of x derived from the equation to the surface by making $z = 0$ and $y = 0$.

If we wish to find the volume of the solid terminated laterally by a cylinder perpendicular to the plane of xy, having for its base any curve as $LL'NN'$ (fig. 54), we take the integral with respect to y from $y = MN$ to $y = MN'$, which are given in terms of x by the equation to the cylinder; and then we integrate with respect to x between the limits of that variable corresponding to the extreme points of the curve which is the base of the cylinder, such as HK and $H'K'$ in the figure. It is to be observed that in getting the limiting values of y in terms of x we introduce into the integral new functions which may often render the formula unintegrable.

(1) To find the volume of the octant of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

We have $V = \iiint dx\,dy\,dz = \iint z\,dx\,dy$,

the limits of z being 0 and its value given in terms of x and y by the equation to the surface, that is,

$$z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}.$$

Therefore $V = c\iint dx\,dy\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}.$

Integrating with respect to y, we have

$$V = \frac{c}{2}\int dx\left\{y\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} + b\left(1 - \frac{x^2}{a^2}\right)\sin^{-1}\frac{ay}{b\,(a^2 - x^2)^{\frac{1}{2}}}\right\}.$$

Now $\int z\,dy$ represents the area of a section parallel to the plane yz, and at a distance equal to x: the integral must therefore be taken between the limits given by that section, or from $y = 0$ to $y = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}}$. This gives

$$V = \frac{\pi}{2}\,\frac{bc}{2}\int dx\left(1 - \frac{x^2}{a^2}\right) = \frac{\pi}{2}\,\frac{bc}{2}\left(x - \frac{x^3}{3a^2}\right) + C.$$

The limits of x are 0 and a, so that we have

$$V = \frac{\pi abc}{6};$$

and the volume of the whole ellipsoid, being eight times this quantity, is $\frac{4}{3}\pi abc$. When $a = b = c$, the ellipsoid becomes a sphere, the volume of which consequently is $\frac{4}{3}\pi a^3$.

(2) The equation to the elliptic paraboloid being

$$\frac{z^2}{a} + \frac{y^2}{b} = 2x,$$

we have $V = \iint z\,dx\,dy = \left(\frac{a}{b}\right)^{\frac{1}{2}}\iint dx\,dy\,(2bx - y^2)^{\frac{1}{2}}$.

On integrating with respect to y from $y = 0$ to $y = (2bx)^{\frac{1}{2}}$, this gives $V = \frac{\pi}{2}(ab)^{\frac{1}{2}}\int x\,dx = \frac{\pi}{4}x^2(ab)^{\frac{1}{2}}$,

which taken from $x = 0$ to $x = c$, and multiplied by 4, gives $\pi c^2(ab)^{\frac{1}{2}}$ as the volume of the paraboloid intercepted between the vertex and a plane parallel to the plane of yz at a distance c.

(3) The equation to the Cono-Cuneus of Wallis is

$$c^2z^2 = y^2(a^2 - x^2);$$

the whole volume is

$$V = \pi a^2 c.$$

(4) The general equation to conical surfaces is (the origin being at the vertex and the axis of x being the axis of the cone)

$$z = x\,\phi\left(\frac{y}{x}\right).$$

Therefore $V = \int\int dx dy\, x\phi\left(\frac{y}{x}\right)$, or putting $y = ax$ and therefore $dy = x da$, we have

$$V = \int\int dx da\, x^2 \phi(a) = \int da \frac{x^3}{3} \phi(a).$$

If the base be a plane curve parallel to yz, the limits of x are 0 and a, so that

$$V = \frac{a^3}{3} \int da \phi(a).$$

Now the equation to the base is found by making $x = a$ in the equation to the surface, which then becomes $z = a\phi\left(\frac{y}{a}\right)$, and, as in this case, $y = a\alpha$, $dy = a d\alpha$,

$$V = \frac{a}{3} \int z dy.$$

Now as z is the ordinate of the base, $\int z dy$ is its area, so that the volume of the cone is one third of its base multiplied into its altitude.

(5) The axes of two equal right circular cylinders intersect at right angles; find the volume of the solid common to both.

Taking the intersection of the axes of the cylinders as the origin, and their axes as the axes of y and z, their equations are

$$x^2 + z^2 = a^2, \quad x^2 + y^2 = a^2,$$

$$V = \int\int dx dy z = \int\int dx dy (a^2 - x^2)^{\frac{1}{2}}.$$

Integrating with respect to y from $y = 0$ to $y = (a^2 - x^2)^{\frac{1}{2}}$,

$$V = \int dx (a^2 - x^2);$$

and integrating with respect to x from $x = 0$ to $x = a$,

$$V = \frac{2a^3}{3}.$$

This is the eighth part of the whole intercepted solid which is therefore $\frac{16a^3}{3}$.

(6) The axis of a right circular cylinder passes through the centre of a sphere, find the volume of the solid which is common to both surfaces.

Taking the centre of the sphere as origin, and the axis of the cylinder as the axis of z, the equations to the surfaces are

$$x^2 + y^2 + z^2 = a^2, \quad x^2 + y^2 = b^2.$$

The direct integration of $\int\int z\, dx dy$ in terms of x and y, leads to operations of considerable complexity which may be avoided by transforming x and y into polar co-ordinates r and θ: in which case by Chap. III. Sect. 2. of the Diff. Calc., we have

$$dxdy = rdrd\theta, \text{ and}$$

$$V = \int\int zrdrd\theta = \int\int drd\theta r(a^2 - r^2)^{\frac{1}{2}};$$

the limits of θ being 0 and 2π; and those of r being 0 and b.

$$\text{Hence} \qquad V = 2\pi \int_0^b dr\, r(a^2 - r^2)^{\frac{1}{2}}$$

$$= \frac{2\pi}{3} \{a^3 - (a^2 - b^2)^{\frac{3}{2}}\}:$$

and the whole volume of the included solid being double this quantity is $\frac{4\pi}{3} \{a^3 - (a^2 - b^2)^{\frac{3}{2}}\}$.

The volume of the sphere is $\frac{4\pi}{3} a^3$, and therefore the volume of the solid intercepted between the concave surface of the sphere and the convex surface of the cylinder, is

$$\frac{4\pi}{3} (a^2 - b^2)^{\frac{3}{2}}.$$

(7) A sphere is cut by a cylinder, the radius of whose base is half of that of the sphere, and whose axis bisects the radius of the sphere at right angles; find the volume of the solid common to both surfaces.

The equations to the surfaces in this case are

$$x^2 + y^2 + z^2 = a^2, \text{ and } x^2 + y^2 = ax.$$

Transforming x and y into polar co-ordinates as in the last example, we have

$$V = \int\int drd\theta rz = \int\int drd\theta\, r\, (a^2 - r^2)^{\frac{1}{2}};$$

the limits of r being 0 and $a\cos\theta$, and those of θ being 0 and $\frac{1}{2}\pi$. Hence

$$V = \frac{a^3}{3}\int_0^{\frac{1}{2}\pi} d\theta\left\{1 - (\sin\theta)^3\right\} = \frac{a^3}{3}\left(\frac{\pi}{2} - \frac{2}{3}\right).$$

Therefore the whole solid cut out from the hemisphere is $\dfrac{2\pi}{3}a^3 - \dfrac{8a^3}{9}$; and the part of the hemisphere which is not comprised in the cylinder is $\dfrac{8a^3}{9}$ or $\dfrac{1}{9}$ of the cube of the diameter of the sphere.

(8) A paraboloid of revolution is pierced by a right circular cylinder, the axis of which passes through the focus and cuts the axis at right angles, its radius being one fourth of the latus rectum of the generating parabola; find the volume of the solid common to the two surfaces.

The equations to the surfaces are

$$y^2 + z^2 = 4ax, \quad x^2 + y^2 = 2ax.$$

Hence
$$V = \iint dx\,dy\,(4ax - y^2)^{\frac{1}{2}}$$
$$= \tfrac{1}{2}\int_0^a dx\left\{x(4a^2 - x^2)^{\frac{1}{2}} + 4ax\sin^{-1}\tfrac{1}{2}\left(\frac{2a-x}{a}\right)^{\frac{1}{2}}\right\}$$
$$= a^3\left(\frac{4}{3} + \frac{\pi}{2}\right);$$

and the whole solid is $a^3\left(\dfrac{16}{3} + 2\pi\right)$.

When a solid is generated by the motion of a plane area which moves parallel to itself, while its magnitude increases or decreases according to a given law, its volume is found by the formula

$$V = \cos\alpha\int v\,dz;$$

v being the area, the axis of z being the direction of motion, and making a constant angle α with the normal to the plane.

(9) Let the solid be the groin which is generated by a square moving parallel to itself, its sides being the double

ordinates of a circle of which z is the abscissa. If y be the half length of a side, $v = 4y^2$, and $y^2 = a^2 - z^2$, and as in this case $\alpha = 0$, we have

$$V = 4\int_0^a dz\,(a^2 - z^2) = \frac{8}{3}a^3.$$

(10) Find the content of the solid *ABCDO* (fig. 55); the base *ABCD* being a rectangle, the side *OAB* a right-angled triangle perpendicular to the plane of the rectangle, and the upper side *OBCD* being formed by drawing lines as *PQ* from *OB* to *CD*, always parallel to the plane *OAD*. If we draw *PR* parallel to *OA*, and join *RQ*, the triangle *PQR* having two sides parallel to the sides of *ODA*, is in a plane parallel to that of *ODA*. Hence the figure may be supposed to be generated by the motion of a triangle constantly parallel to *AOD*, and having its angular points in the lines *AB*, *OB*, *CD*. If $AD = a$, $AB = b$, $AO = c$, the volume of the solid is $\frac{abc}{4}$.

(11) The axes of two equal right circular cylinders intersect at an angle α, to find the volume of the solid common to both.

Let *ABCD* (fig. 56) be the section of the solid made by the plane containing the axes, and let the radius of the cylinders $= a$, so that $AB = a \operatorname{cosec} \alpha$.

If we cut the solid by a plane parallel to *ABCD*, we shall have a parallelogram as *PQRS*; and calling the area of this A, and its distance from the plane of the axes z, we shall have for the part of the solid above that plane

$$V = \int_0^a dz\,A.$$

Now $A = 4POQ$; but making $PQ = l$, and calling p the perpendicular on PQ from the point in the plane *PQRS* where it meets a line through O perpendicular to the plane of the axes, we have

$$l = p\,(\tan\tfrac{1}{2}\alpha + \cot\tfrac{1}{2}\alpha) = 2p \operatorname{cosec} \alpha,$$

and therefore $POQ = p^2 \operatorname{cosec} \alpha$, and $A = 4p^2 \operatorname{cosec} \alpha$. But the section through O and the perpendicular p being a semicircle, we have $p^2 = a^2 - z^2$. Hence

$$V = 4 \operatorname{cosec} \alpha \int_0^a (a^2 - z^2)\, dz = \frac{8}{3} a^3 \operatorname{cosec} \alpha,$$

and therefore the whole solid is $\dfrac{16a^3}{3 \sin \alpha}$.

(12) Find the volume of the solid $DEQB$ (fig. 57) cut off from a right circular cylinder by a plane EQD passing through the centre of the base, and inclined at an angle α to the plane of the base.

If we cut the solid by a plane perpendicular to the base of the cylinder, and parallel to the trace ED, the section is a parallelogram, and the solid may be considered as generated by the motion of this parallelogram parallel to itself.

Let $CB = a$, $CM = x$, $MN = y$, $PN = z$, then

$$\text{as } z = x \tan \alpha, \text{ and } y = (a^2 - x^2)^{\frac{1}{2}},$$

$$V = 2 \tan \alpha \int dx\, x\, (a^2 - x^2)^{\frac{1}{2}}$$

$$= \tan \alpha \left\{C - \tfrac{2}{3}(a^2 - x^2)^{\frac{3}{2}}\right\}.$$

When $x = 0$, $V = 0$, therefore $C = a^3$; hence

$$V = \tfrac{2}{3} \tan \alpha \left\{a^3 - (a^2 - x^2)^{\frac{3}{2}}\right\};$$

and the whole solid when $x = a$ is $\frac{2}{3} a^3 \tan \alpha$.

If the solid be one of revolution round the axis of x, and if $y = f(x)$ be the equation to the generating curve, the volume of the solid is given by the integral $V = \pi \int y^2 dx$.

(13) A paraboloid formed by the revolution of a parabola round its axis.

In this case $y^2 = 4mx$, and

$$V = 4m\pi \int x\, dx = 2m\pi x^2 + C.$$

If the solid be reckoned from the vertex $C = 0$, and

$$V = 2m\pi x^2 = \tfrac{1}{2} \pi y^2 x.$$

(14) The volume of an oblate spheroid formed by the revolution of an ellipse round its minor axis is $\dfrac{4\pi a^2 b}{3}$, a

being the major axis of the ellipse; and the volume of a prolate spheroid is $\frac{4\pi a b^2}{3}$.

(15) Find the volume of the solid formed by the revolution of the cissoid round its asymptote.

The asymptote being taken as the axis of x, the equation to the cissoid is

$$x^2 y = (2a - y)^3,$$

a being the radius of the generating circle.

$$\text{Now} \quad V = \pi \int dx\, y^2 = \pi \int dy\, y^2 \frac{dx}{dy},$$

$$= -\tfrac{1}{2}\pi \int dy \left\{3y^{\frac{3}{2}}(2a - y)^{\frac{1}{2}} + y^{\frac{1}{2}}(2a - y)^{\frac{3}{2}}\right\}$$

$$= -\pi \int dy\, (a + y)(2ay - y^2)^{\frac{1}{2}}$$

$$= C + \tfrac{1}{3}\pi (2ay - y^2)^{\frac{3}{2}} - 2a\pi \,.\, \text{circ. area whose vers.} = \frac{y}{a}.$$

When $y = 2a$, $V = 0$, therefore $C = \pi^2 a^3$, and

$$V = \pi^2 a^3 + \tfrac{1}{3}\pi (2ay - y^2)^{\frac{3}{2}} - 2a\pi \,.\, \text{circ. area whose vers.} = \frac{y}{a}.$$

Hence the whole volume is $2\pi^2 a^3$.

(16) The equation to the conchoid being

$$xy = (a + y)(b^2 - y^2)^{\frac{1}{2}},$$

the volume formed by its revolution round the axis of x is

$$V = \frac{\pi^2 a b^2}{2} - \pi a b^2 \sin^{-1}\frac{y}{b} + \frac{\pi}{3}(b^2 - y^2)^{\frac{1}{2}}(y^2 + 2b^2),$$

and the whole volume is $\pi b^2 \left(\pi a + \frac{4b}{3}\right)$.

(17) The equation to the cycloid is (the base being the axis of x),

$$\frac{dy}{dx} = \left(\frac{2a - y}{y}\right)^{\frac{1}{2}}.$$

Therefore the volume formed by its revolution round the base is

$$V = \pi \int dy\, y^2 \frac{dx}{dy} = \pi \int dy \frac{y^3}{(2ay - y^2)^{\frac{1}{2}}}.$$

This being integrated from $y = 0$ to $y = 2a$ and doubled gives $5\pi^2 a^3$ as the volume of the whole solid.

When the axis of the cycloid is taken as the axis of x, the equation to the curve is

$$\frac{dy}{dx} = \left(\frac{2a - x}{x}\right)^{\frac{1}{2}};$$

but this is not a convenient form for finding the value of $\pi \int y^2 \, dx$. It is better to substitute for y and x their expressions in terms of θ, i. e.

$$y = a(\theta + \sin\theta), \quad x = a(1 - \cos\theta);$$

whence $V = \pi a^3 \int d\theta \sin\theta \, (\theta + \sin\theta)^2$.

The value of this taken from $\theta = 0$ to $\theta = \pi$, is

$$V = \pi a^3 \left(\frac{3\pi^2}{2} - \frac{8}{3}\right).$$

(18) The equation to the tractory is

$$(a^2 - y^2)^{\frac{1}{2}} \frac{dy}{dx} + y = 0;$$

and the volume of the solid generated by its revolution round the axis of x, and taken from $x = 0$ to $x = \infty$ is $\frac{1}{3}\pi a^3$.

(19) The equation to the Witch of Agnesi is

$$xy = 2a(2ay - y^2)^{\frac{1}{2}}.$$

If it revolve round its asymptote which is taken as the axis of x, we have for the volume of the solid

$$V = \pi \int y^2 \, dx = \pi y^2 x - 2\pi \int xy \, dy$$
$$= \pi y^2 x - 4\pi a \int dy \, (2ay - y^2)^{\frac{1}{2}}.$$

The whole volume is $4\pi^2 a^3$.

(20) The companion of the cycloid is defined by the equations

$$y = a\theta, \quad x = a(1 - \cos\theta).$$

The volume of the solid generated by its revolution round the axis of y, or the base of the curve is

$$V = \pi \int dy \, (2a - x)^2 = \pi a^3 \int d\theta \, (1 + \cos\theta)^2;$$

which taken from $\theta=0$ to $\theta=\pi$, and doubled, gives as the whole volume of the solid generated

$$V=3\pi^2a^3.$$

If the curve revolve round the axis of x,

$$V=\pi\int y^2\,dx=\pi a^3\int d\theta\,\theta^2\sin\theta;$$

which taken from $\theta=0$ to $\theta=\pi$, gives as the volume of the whole solid generated

$$V=\pi(\pi^2-4)a^3.$$

SECT. 4. *Quadrature of Surfaces.*

The general expression for the surface of a solid is

$$S=\iint dx\,dy\left\{1+\left(\frac{dz}{dx}\right)^2+\left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}},$$

the limits of x and y being determined as in the cubature of solids.

(1) In the sphere where $x^2+y^2+z^2=r^2$, we have

$$\frac{dz}{dx}=-\frac{x}{z},\quad \frac{dz}{dy}=-\frac{y}{z},$$

so that $$S=r\iint\frac{dx\,dy}{(r^2-x^2-y^2)^{\frac{1}{2}}}.$$

Integrating with respect to y, we have

$$S=r\int dx\sin^{-1}\frac{y}{(r^2-x^2)^{\frac{1}{2}}},$$

which when taken from $y=0$ to $y=(r^2-x^2)^{\frac{1}{2}}$ gives

$$S=\tfrac{1}{2}\pi rx+C,$$

and, supposing S to vanish when $x=0$,

$$S=\tfrac{1}{2}\pi rx.$$

This is the part of the surface included within the positive axes, and if we multiply it by 4 we have $2\pi rx$ as the surface of a zone of the sphere, the height of which is x: it is therefore equal to the corresponding zone of the circumscribing cylinder. The whole surface of the sphere is $4\pi r^2$ or four times the area of a great circle.

(2) The axes of two equal right circular cylinders intersect at right angles, find the area of the surface of the one which is intercepted by the other. The equations are

$$x^2 + z^2 = a^2, \quad x^2 + y^2 = a^2;$$

and $$S = \iint dx\,dy \left\{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2\right\}^{\frac{1}{2}}.$$

Here $$\frac{dz}{dx} = -\frac{x}{z}, \quad \frac{dz}{dy} = 0;$$

therefore $$S = a\iint \frac{dx\,dy}{z} = a\iint \frac{dx\,dy}{(a^2 - x^2)^{\frac{1}{2}}}.$$

Integrating with respect to y from $y = 0$ to $y = (a^2 - x^2)^{\frac{1}{2}}$,

$$S = a\int_0^a dx = a^2;$$

and the whole surface, being eight times this, is $8a^2$.

(3) Circumstances being the same as in Ex. (7) of the last section, to find the area of the intercepted surface of the sphere. The equations to the surfaces being

$$x^2 + y^2 + z^2 = a^2, \quad x^2 + y^2 = ax,$$

$$S = a\iint \frac{dx\,dy}{z} = a\iint \frac{dx\,dy}{(a^2 - x^2 - y^2)^{\frac{1}{2}}}.$$

Transforming into polar co-ordinates r and θ, we have

$$S = a\iint \frac{r\,dr\,d\theta}{(a^2 - r^2)^{\frac{1}{2}}};$$

the limits of r being 0 and $a \cos\theta$, those of θ being 0 and $\frac{1}{2}\pi$. Therefore

$$S = a^2 \int_0^{\frac{1}{2}\pi} (1 - \sin\theta) = a^2\left(\tfrac{1}{2}\pi - 1\right).$$

The area of the surface of the octant of the sphere is $\frac{1}{2}\pi a^2$; and therefore the area of the surface of the octant which is not included in the cylinder is equal to a^2, or the square of the radius of the sphere. If the sphere be pierced by two equal and similar cylinders, the area of the non-intercepted surface is $8a^2$, or twice the square of the diameter of the sphere.

This is the celebrated Florentine enigma which was proposed by Vincent Viviani as a challenge to the mathematicians of his day in the following form:

> "Inter venerabilia olim Græciæ monumenta extat adhuc, perpetuo quidem duraturum, Templum augustissimum ichnographia circulari ALMÆ GEOMETRIÆ dicatum, quod testudine intus perfecte hemisphærica operitur: sed in hac fenestrarum quatuor æquales areæ (circum ac supra basin hemisphæræ ipsius dispositarum) tali configuratione, amplitudine, tantaque industria, ac ingenii acumine sunt exstructæ, ut his detractis superstes curva Testudinis superficies, pretioso opere musivo ornata, tetragonismi vere geometrici sit capax.
>
> *Acta Eruditorum*, 1692.

(4) Under the same circumstances to find the area of the intercepted surface of the cylinder.

The element of the circumference of the base of the cylinder being $\frac{a}{2}\frac{dx}{(ax-x^2)^{\frac{1}{2}}}$, we have

$$S = \frac{a}{2}\int_0^a \frac{z\,dx}{(ax-x^2)^{\frac{1}{2}}} = \frac{a^{\frac{3}{2}}}{2}\int_0^a \frac{dx}{x^{\frac{1}{2}}} = a^2,$$

and the whole area of the intercepted surface of the cylinder is $4a^2$, or equal to the square of the diameter of the sphere.

If a solid be generated by the motion of a plane parallel to itself, the surface may be found by a method similar to that used for finding the volume. If u be the periphery of the generating plane, s the arc of the curve made by a plane perpendicular to the generating plane

$$S = \int u\,ds.$$

(5) Under the same circumstances as in Ex. (11) of the last section, to find the surface of the intercepted solid.

If s be the arc of the circle passing through O and perpendicular to PQ, the area of the element $PQpq$ is $l\,ds$, and the area of the surface AOB is $\int l\,ds$.

Now $\quad l = 2p \operatorname{cosec} \alpha = 2(a^2 - z^2)^{\frac{1}{2}} \operatorname{cosec} \alpha,$

$$\text{and} \quad ds = \frac{a\,dz}{(a^2 - z^2)^{\frac{1}{2}}};$$

$$\text{therefore} \quad S = 2a \operatorname{cosec} \alpha \int_0^a dz = 2a^2 \operatorname{cosec} \alpha;$$

and the whole surface is $16a^2 \operatorname{cosec} \alpha$.

(6) Under the same circumstances as in Ex. (12) of the last section to find the area of the convex surface of the part of the cylinder cut off.

s being the element of the circumference of the base, and S the element of the surface.

$$S = 2\int z\,ds = 2a \tan \alpha \int \frac{x\,dx}{(a^2 - x^2)^{\frac{1}{2}}}$$

$$= 2a \tan \alpha \left\{C - (a^2 - x^2)^{\frac{1}{2}}\right\}.$$

When $x = 0$, $S = 0$ and $C = a$, therefore

$$S = 2a \tan \alpha \left\{a - (a^2 - x^2)^{\frac{1}{2}}\right\};$$

and the whole convex surface is $2a^2 \tan \alpha$.

When a curve surface is formed by the revolution round the axis of x of a curve the equation to which is $y = f(x)$, the area of the surface is given by the integral,

$$S = 2\pi \int dx\, y \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

(7) For the paraboloid of revolution we have

$$y^2 = 4mx;$$

$$\text{therefore} \quad S = 4\pi m^{\frac{1}{2}} \int dx\,(x + m)^{\frac{1}{2}}$$

$$= \frac{8\pi}{3} m^{\frac{1}{2}} (x + m)^{\frac{3}{2}} + C.$$

If the surface be measured from the origin,

$$S = \frac{8\pi}{3} m^{\frac{1}{2}} \left\{(x + m)^{\frac{3}{2}} - m^{\frac{3}{2}}\right\}.$$

(8) For the prolate spheroid we have

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2);$$

and $S = \frac{2\pi b}{a} \int dx\, (a^2 - e^2 x^2)^{\frac{1}{2}}$, where $e^2 = \frac{a^2 - b^2}{a^2}$.

Integrating, we have

$$S = \frac{\pi a b}{e} \left\{ \sin^{-1} \frac{e x}{a} + \frac{e x}{a} \left(1 - \frac{e^2 x^2}{a^2} \right)^{\frac{1}{2}} \right\} + C.$$

If the whole surface be required, this is to be taken from $x = -a$ to $x = +a$, so that

$$S = \frac{2\pi a b}{e} \{ \sin^{-1} e + e (1 - e^2)^{\frac{1}{2}} \}.$$

In the oblate spheroid we have for the whole surface

$$S = 2\pi a^2 \left\{ 1 + \frac{1 - e^2}{2e} \log_e \left(\frac{1 + e}{1 - e} \right) \right\}.$$

(9) The equation to the cycloid (the base being the axis of x) is

$$\frac{dy}{dx} = \left(\frac{2a - y}{y} \right)^{\frac{1}{2}};$$

therefore $S = 2\pi \int dx\, (2ay)^{\frac{1}{2}} = 2\pi \int dy\, (2ay)^{\frac{1}{2}} \frac{dx}{dy}$

$$= 2\pi (2a)^{\frac{1}{2}} \int dy \frac{y}{(2a - y)^{\frac{1}{2}}}.$$

Integrating this and extending the integral over the whole surface, we find

$$S = \frac{64}{3} \pi a^2.$$

When the axis of the curve is taken as the axis of x, and therefore of revolution, the equation to the curve is

$$\frac{dy}{dx} = \left(\frac{2a - x}{x} \right)^{\frac{1}{2}}.$$

Here $S = 2\pi \int dx\, y \left(\frac{2a}{x} \right)^{\frac{1}{2}}$, and integrating by parts,

$$S = 4\pi y (2ax)^{\frac{1}{2}} - 4\pi (2a)^{\frac{1}{2}} \int dx\, (2a - x)^{\frac{1}{2}}.$$

Integrating from $x = 0$ to $x = 2a$, we have for the whole surface

$$S = 8\pi a^2 \left(\pi - \frac{4}{3}\right).$$

(10) The surface of the solid generated by the revolution of the tractory

$$\frac{dy}{dx} + \frac{y}{(a^2 - y^2)^{\frac{1}{2}}} = 0,$$

round the axis of x, and taken from $x = 0$ to $x = \infty$ is equal to $2\pi a^2$.

CHAPTER X.

GEOMETRICAL PROBLEMS INVOLVING THE SOLUTION OF DIFFERENTIAL EQUATIONS.

QUESTIONS of this kind were by the early writers on the Differential Calculus called Problems in the Inverse Method of Tangents, because, as the direct processes of the Differential Calculus were originally invented for the purpose of drawing tangents to curves, so the inverse Calculus had for its object the investigation of the equations of curves from the properties of their tangents and lines connected with them.

(1) Let it be required to find the curve in which the subtangent is a multiple of the abscissa.

If $$y=f(x)$$

be the equation to the curve, $y\dfrac{dx}{dy}$ is the subtangent. Therefore we have the condition

$$y\frac{dx}{dy}=mx, \quad \text{or} \quad \frac{dx}{x}=m\frac{dy}{y},$$

$$\text{whence} \quad \log x = m\log y + C = \log Cy^m,$$

$$\text{and} \quad x = Cy^m.$$

When m is positive this gives a parabola of the m^{th} order; when m is negative it gives a hyperbola of the same order.

(2) Find the curve in which the area contained between the axis of x, the ordinate and the curve, is a multiple of the rectangle contained by the ordinate and the abscissa. This stated analytically gives the equation

$$\int y\,dx=\frac{m}{n}xy, \quad \text{or} \quad y\,dx=\frac{m}{n}(x\,dy+y\,dx).$$

The integral of this is

$$x^{n-m} = Cy^m.$$

When $n = 3$, $m = 2$, this gives the common parabola, as is otherwise obvious.

(3) Find the curve in which the perpendicular from the origin on the tangent is equal to the abscissa.

The differential equation is

$$y - x\frac{dy}{dx} = x\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}};$$

$$\text{or}\quad y^2 - x^2 = 2xy\frac{dy}{dx}.$$

This is a homogeneous equation, and on being integrated it gives

$$y^2 + x^2 = cx,$$

the equation to a circle, the origin being in the circumference and the axis of x being a diameter.

(4) Find the curve in which the distance from the origin is equal to the part of the tangent intercepted between the point of contact and the perpendicular from the origin.

The differential equation is

$$ydx - xdy = ydy + xdx;$$

and the integral is

$$\log\frac{(x^2 + y^2)}{C} = \tan^{-1}\left(\frac{y}{x}\right),$$

which is the equation to a logarithmic spiral, the constant angle of which is equal to $\frac{\pi}{4}$.

(5) Find the nature of the curve BP (fig. 58.) such that, if from the origin A a line AQ be drawn making an angle of 45° with the axis of x, and meeting the ordinate at any point P in Q, the ordinate PM shall bear to the sub-tangent MT the same ratio which the difference between PM and MQ bears to a constant line (a).

The sub-tangent $= y\dfrac{dx}{dy}$, and $QM = AM = x$; therefore

$$y : y\frac{dx}{dy} = y - x : a,$$

or $$(y - x)\,dx = a\,dy, \text{ is the equation.}$$

This, when put under the form

$$a\,dy - y\,dx + x\,dx = 0,$$

is a linear equation of the first order, and its integral is

$$y = x + a + C\epsilon^{\frac{x}{a}}.$$

Since when $y = 0$, the curve must pass through the origin, we have $C = -a$, and therefore

$$y = x + a - a\epsilon^{\frac{x}{a}}.$$

This curve at one time attracted much attention, and it appears to have been the first problem involving a differential equation which was solved. It was proposed to Descartes* by De Beaune, after whom the curve is usually called "Curva Beauniana." The solution will be found in the works of John Bernoulli, Vol. I. p. 63, and p. 65.

(6) Find the curve in which the product of perpendiculars from two fixed points on the tangent is constant.

Let A, B (fig. 59.) be the fixed points; take C the middle point between them as origin, and the line joining them as the axis of x. Then x and y being the co-ordinates of the point of contact P, the perpendiculars AY and BZ are, if $CA = CB = c$,

$$\frac{y - (x - c)\dfrac{dy}{dx}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}} \quad \text{and} \quad \frac{y - (x + c)\dfrac{dy}{dx}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}.$$

Hence making their product constant and equal to b^2, we have

$$\left(y - x\frac{dy}{dx}\right)^2 - c^2\left(\frac{dy}{dx}\right)^2 = b^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}.$$

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* See his *Letters*, Tom. III. No. 79.

Differentiating this we get

$$\frac{d^2y}{dx^2} = 0, \text{ and } \{x^2 - (b^2 + c^2)\} \frac{dy}{dx} = xy.$$

The first solution gives the general integral

$$y = ax + a'.$$

Substituting the value of $\frac{dy}{dx}$ derived from this in the given equation, we have

$$y - ax = \pm \{b^2 + a^2 (b^2 + c^2)\},$$

the equation to two straight lines.

The second solution by the elimination of $\frac{dy}{dx}$ gives the singular solution, which is

$$\frac{y^2}{b^2} + \frac{x^2}{b^2 + c^2} = 1$$

the equation to an ellipse, the minor axis of which is equal to b.

Euler, *Mémoires de Berlin*, 1756.

(7) Find the curve in which the normal bears a constant ratio to its intercept on the axis of x.

The length of the normal is $y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$;

that of the intercept is $x + y\frac{dy}{dx}$,

and if n be the constant ratio, we have

$$y\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = n\left(x + y\frac{dy}{dx}\right).$$

Squaring, and solving with respect to $y\frac{dy}{dx}$, we have

$$(n^2 - 1)\, y\frac{dy}{dx} + n^2x = \pm \{(n^2 - 1)\, y^2 + n^2x^2\}^{\frac{1}{2}},$$

whence $\quad \{(n^2 - 1)\, y^2 + n^2x^2\}^{\frac{1}{2}} = C \pm x,$

or $(n^2 - 1)(y^2 + x^2) = C^2 \pm 2Cx,$

which is the equation to a series of circles.

When $n < 1$, the equation to the locus of the ultimate intersection of these is

$$(1 - n^2)^{\frac{1}{2}} y \pm nx = 0,$$

giving two straight lines which obviously satisfy the condition.

(8) Find the curve in which the area is equal to the cube of the ordinate divided by the abscissa.

The condition is

$$\int y\,dx = \frac{y^3}{x},$$

whence we have

$$(x^2 y + y^3)\,dx = 3xy^2 dy.$$

This being homogeneous may be integrated in the usual way—the result is

$$x^2 - 2y^2 = Cx^{\frac{2}{3}}.$$

(9) Find the curve in which the normal is equal to the distance from the origin.

It is a circle or an equilateral hyperbola according as the two lines are on the same or opposite sides of the curve.

Trajectories.

A Trajectory may be defined to be a line which cuts, according to a given law, a series of curves expressed by one equation. Problems of this kind, at least with respect to Trajectories cutting curves at a constant angle were first proposed by John Bernoulli*, but they were also considered by several other writers, as James Bernoulli†, and Leibnitz, who in 1715 proposed it as a challenge to English Analysts, "ad pulsum Anglorum Analystorum nonnihil tentandum;" a challenge which was answered by Newton‡ and by Taylor‖.

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* *Com. Epis. Leib. et Bern.* Vol. I. p. 17, and *Opera*, Var. Loc.

† *Opera*, Var. Loc.

‡ *Phil. Trans.* 1716.

‖ *Id.* 1717.

The most elaborate discussion however of the problem is by Euler in three memoirs in the *Novi Commen. Petrop.* Vol. XIV. p. 46, Vol. XVII. p. 205; and *Nova Acta Petrop.* Vol. I. p. 3.

Let $F(x', y', a) = 0$ be the equation to a series of curves which are to be cut by a trajectory at a constant angle: then if $f(xy) = 0$ be the equation to the proposed trajectory, and c be the tangent of the angle between the two curves, we have the condition

$$\frac{\dfrac{dy'}{dx'} - \dfrac{dy}{dx}}{1 + \dfrac{dy'}{dx'}\dfrac{dy}{dx}} = c.$$

The differential equation to the trajectory is found by eliminating a between the equation to the curves, and this last equation, where, it is to be observed, x and y are to be substituted for x' and y' in the value of $\dfrac{dy'}{dx'}$. If the trajectory is to be rectangular, since in that case $c = \infty$, the condition becomes

$$1 + \frac{dy'}{dx'}\frac{dy}{dx} = 0.$$

The elimination of a may be more or less easily effected according to the way in which it is involved in the given equation to the series of curves: if that can be put under the form

$$\phi(x, y) = a,$$

the elimination is readily effected.

(10) To find the curve which cuts at a constant angle a series of straight lines drawn from one point.

That point being taken as origin, the equation to the lines is

$$y' = ax' \quad \text{or} \quad \frac{y'}{x'} = a,$$

a being the variable parameter. The equation of condition becomes

$$\frac{dy}{dx} - a = c\left(1 + a\frac{dy}{dx}\right).$$

But
$$a = \frac{y'}{x'} = \frac{y}{x};$$

therefore $\quad x\,dy - y\,dx = c\,(y\,dy + x\,dx).$

Dividing both sides by $x^2 + y^2$,

$$\frac{x\,dy - y\,dx}{x^2 + y^2} = c\,\frac{y\,dy + x\,dx}{x^2 + y^2}.$$

Integrating,

$$\tan^{-1}\frac{y}{x} = c\log(x^2 + y^2)^{\frac{1}{2}} + a.$$

If we make $\dfrac{y}{x} = \tan\theta$, $x^2 + y^2 = r^2$, this may be put under the form

$$r = b\,e^{\frac{\theta}{c}},$$

which is the equation to the logarithmic spiral.

(11) Let the given curves be all the circles which pass through one point at which they all touch one straight line. Taking this line as the axis of y, the equation to the circles is

$$y'^2 = 2ax' - x'^2 \quad \text{or} \quad a = \frac{x'^2 + y'^2}{2x'}.$$

Then $\dfrac{dy'}{dx'} = -\dfrac{(x'^2 - y'^2)}{2x'y'}$, and the equation of condition is

$$\{c\,(x^2 - y^2) + 2xy\}\,dy + (x^2 - y^2 - 2cxy)\,dx = 0.$$

This being a homogeneous equation may be integrated by the appropriate method, and the result is

$$x^2 + y^2 = b\,(cy - x),$$

b being the arbitrary constant introduced in the integration. This is evidently the equation to a circle.

(12) Find the orthogonal trajectory of the series of parabolas represented by the equation

$$y'^2 = 4ax'.$$

The equation of condition for orthogonal trajectories is

$$1 + \frac{dy}{dx}\frac{dy'}{dx'} = 0.$$

In this case $\frac{dy'}{dx'} = \frac{y'}{2x'} = \frac{y}{2x}$; therefore

$$2x\,dx + y\,dy = 0,$$

whence, by integration,

$$x^2 + \frac{y^2}{2} = b^2,$$

b^2 being an arbitrary constant. This is evidently the equation to an ellipse.

The equation to the lemniscate of Bernoulli is

$$(x^2 + y^2)^2 = a^2(x^2 - y^2).$$

That to the orthogonal trajectory is

$$(x^2 + y^2)^2 = bxy;$$

which is the equation to a similar lemniscate, the axis of which is inclined at an angle of 45° to that of the former.

If one of the variables be given as a function of the other and the parameter, as if

$$y' = f(x', a),$$

we cannot eliminate a so readily. But let

$$dy' = Pdx' + Qda;$$

then for one curve $\frac{dy'}{dx'} = P$, and the equation to the orthogonal trajectories is

$$dx + Pdy = 0;$$

and as $dy = Pdx + Qda$, this becomes

$$(1 + P^2)\,dx + PQda = 0.$$

As P and Q contain only x and a, this is an equation between two variables x and a, and to integrate it appropriate methods must be employed.

(14) Let the curves be a series of ellipses expressed by the equation

$$y' = \frac{a}{c}(c^2 - x'^2)^{\frac{1}{2}}.$$

a being the variable parameter. Here

$$P = \frac{-ax'}{c(c^2 - x'^2)^{\frac{1}{2}}}, \quad Q = \frac{(c^2 - x'^2)^{\frac{1}{2}}}{a},$$

and the equation for the orthogonal trajectory is

$$ax(c^2 - x^2)\,da = \{c^4 + (a^2 - c^2)\,x^2\}\,dx.$$

To integrate this put $(c^2 - x^2)^{\frac{1}{2}} = u$, when it becomes

$$au^2da + a^2u\,du = -\frac{c^2u^3du}{c^2 - u^2}.$$

Integrating, substituting for u its value, and eliminating a, the final equation is

$$y^2 = b^2 - x^2 + c^2\log x^2,$$

where b^2 is the arbitrary constant.

When the curves to be intersected are given by a differential equation, the trajectory can be investigated only under particular circumstances. For a detail of these the reader is referred to the memoirs of Euler quoted above.

Involutes of curves may be considered as Orthogonal Trajectories, since they cut at right angles all the tangents of the evolute.

Let $$v = f(u)$$

be the equation to the evolute. The equation to a tangent at any point is

$$y - v = \frac{dv}{du}(x - u), \quad \text{or} \quad y = xw + v - uw,$$

if we put $$w = \frac{dv}{du}.$$

This, then, is the equation to a series of lines to which we wish to find the trajectory. The variable parameter may be considered to be u, of which v and w are functions. Differentiating this equation we have

$$dy = w\,dx + (x - u)\,dw:$$

the general equation to the orthogonal trajectories will then be

$$(1 + w^2)\,dx + (x - u)\,w\,dw = 0.$$

Dividing by $(1 + w^2)^{\frac{1}{2}}$ and transposing, this gives

$$\frac{xw\,dw}{(1 + w^2)^{\frac{1}{2}}} + (1 + w^2)^{\frac{1}{2}}\,dx = \frac{uw\,dw}{(1 + w^2)^{\frac{1}{2}}}.$$

Integrating we find

$$x\,(1 + w^2)^{\frac{1}{2}} = u\,(1 + w^2)^{\frac{1}{2}} - \int du\,(1 + w^2)^{\frac{1}{2}} + C.$$

If the integration on the second side can be accomplished, x is known in terms of u, and then y may also be expressed in terms of the same quantity by means of the equation to the tangent. By eliminating u between these equations we can find the equation to the involute.

(15) Let the curve be the semicubical parabola, the equation to which is

$$27av^2 = 4u^3.$$

Hence $\int du(1 + w^2)^{\frac{1}{2}} = \int du\left(1 + \frac{u}{3a}\right)^{\frac{1}{2}} = 2a\left(1 + \frac{u}{3a}\right)^{\frac{3}{2}};$

therefore $x\left(1 + \frac{u}{3a}\right)^{\frac{1}{2}} = u\left(1 + \frac{u}{3a}\right)^{\frac{1}{2}} - 2a\left(1 + \frac{u}{3a}\right)^{\frac{3}{2}} + C.$

The particular involute is determined by the constant C. Let $x = 2a$ when $u = 0$, then $C = 0$, and therefore

$$x = u - 2a\left(1 + \frac{u}{3a}\right) = \frac{u}{3} - 2a,$$

and $y = -\dfrac{4u^2}{9v}$,

whence, eliminating u and v with the assistance of the given equation, we find

$$y^2 = 4a\,(x + 2a),$$

the equation to a parabola.

(16) If the equation to the evolute be

$$v^{\frac{2}{3}} - u^{\frac{2}{3}} = -c^{\frac{2}{3}};$$

and if the constant be determined by the condition that, when $u = c$, $x = \frac{1}{2}c$, the equation to the involute is

$$y^2 - x^2 = -\frac{1}{4}c^2,$$

shewing that it is an equilateral hyperbola.

If the trajectory is to cut the curves according to any other law than that of a constant angle a similar method is to be employed.

(17) Let, for example, it be required to find the curve $PP'P''$ (fig. 60.) which cuts a series of parabolas having the same axis and the same vertex so that the areas AMP, $AM'P'$, &c. are constant. The equation to the parabola being

$$y^2 = 4ax,$$

the area $APM = \int_0^x y\,dx = 2\int_0^x (ax)^{\frac{1}{2}}\,dx = b^2$ suppose.

Differentiate considering x and a as variables; then

$$2(ax)^{\frac{1}{2}}\,dx + \int_0^x \frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}\,dx\,.\,da = 0,$$

or $$2(ax)^{\frac{1}{2}}\,dx + \frac{da}{a}\int_0^x (ax)^{\frac{1}{2}}\,dx = 0.$$

But by the condition of the area being constant

$$\int_0^x (ax)^{\frac{1}{2}}\,dx = b^2,$$

so that the equation may be put under the form

$$2\,x^{\frac{1}{2}}\,dx + \frac{b^2}{2}\,\frac{da}{a^{\frac{3}{2}}} = 0.$$

Integrating, we have

$$\frac{4}{3}x^{\frac{3}{2}} - \frac{b^2}{a^{\frac{1}{2}}} = \frac{C}{2}.$$

Eliminating a by means of the equation to the parabola,

$$\frac{2}{3}x^{\frac{3}{2}} - \frac{b^2 x^{\frac{1}{2}}}{y} = \frac{C}{4},$$

or
$$y = \frac{b^2 x^{\frac{1}{2}}}{\frac{2}{3} x^{\frac{3}{2}} - \frac{C}{4}}.$$

To determine the arbitrary constant, we observe that when x is indefinitely diminished, y must be indefinitely increased in order that the area may remain constant; this makes $C = 0$. Hence

$$y = \frac{3b^2}{2x}, \qquad \text{or} \quad 2xy = 3b^2,$$

is the required equation, being that to an equilateral hyperbola.

(18) Find the curve which cuts a series of circles described round the same centre in such a way that the arcs intercepted between the intersections and the axis of x shall be equal.

If
$$x^2 + y^2 = a^2$$
be the equation to the circle, and b be the constant length of the arcs, we find, as in the previous example,

$$\frac{a\,dx - x\,da}{(a^2 - x^2)^{\frac{1}{2}}} + \frac{b\,da}{a} = 0.$$

Dividing by a, and integrating

$$\sin^{-1}\left(\frac{x}{a}\right) - \frac{b}{a} = C.$$

Eliminating a, we find

$$\tan^{-1}\frac{x}{y} - \frac{b}{(x^2 + y^2)^{\frac{1}{2}}} = C.$$

By the consideration that when $x = \infty$, y is finite, we determine C to be equal to $\frac{1}{2}\pi$. Hence putting $\frac{y}{x} = -\tan\theta$, and $x^2 + y^2 = r^2$, we have for the equation to the trajectory,

$$r\theta = b,$$

which is the equation to the hyperbolic spiral.

Lagrange[*] in two memoirs has considered the problem of orthogonal trajectories to curve surfaces.

The equation of condition in this case is

$$1 + p\frac{dz}{dx} + q\frac{dz}{dy} = 0,$$

where $p = \frac{dz'}{dx'}$ and $q = \frac{dz'}{dy'}$ in the equation to the surfaces.

By eliminating the constant parameter between this equation and the equation to the surfaces, we obtain a partial differential equation, the integral of which gives the orthogonal trajectory[†].

(19) Let the problem be, to find the surface which cuts at right angles all the spheres which pass through a given point, and have their centres on a given line.

Taking the given point as the origin, and the axis of x as the given line, the equation to the spheres is

$$x'^2 + y'^2 + z'^2 = 2ax'.$$

Hence $$p = \frac{y^2 + z^2 - x^2}{2xz}, \quad q = -\frac{y}{z};$$

and the equation of condition becomes

$$1 + \frac{y^2 + z^2 - x^2}{2xz}\frac{dz}{dx} - \frac{y}{z}\frac{dz}{dy} = 0.$$

To integrate this we have the two equations,

$$\frac{y}{z}dz - dy = 0, \quad \frac{y^2 + z^2 - x^2}{z}dz + 2x\,dx = 0.$$

The first gives $$y = az:$$

the second, on dividing by z, gives

$$\frac{2x}{z}dx - \frac{x^2}{z^2}dz + (1 + a^2)\,dz = 0.$$

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[*] *Berlin Memoirs*, 1779, p. 152, and 1785, p. 176.

[†] There is a Memoir on this subject by Euler in the *Petersburg Memoirs*, Vol. VII., the date of which is 1782, but it was not published till 1820.

Whence $\frac{x^2}{z} + (1 + a^2)z = \beta = \phi(a) = \phi\left(\frac{y}{x}\right)$.

Therefore $$x^2 + y^2 + z^2 = z\phi\left(\frac{y}{z}\right)$$

is the equation to the trajectory. As this contains an arbitrary function, it appears that there is a whole class of surfaces which possess the required property; if we wish to limit them to the surfaces of the second order, we must assume

$$\phi\left(\frac{y}{z}\right) = b\left(\frac{y}{z}\right) + c,$$

in which case the equation becomes

$$x^2 + y^2 + z^2 = by + cz,$$

representing spheres having their centres in the plane of yz.

(20) Find the surface which cuts at right angles all the ellipsoids represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The differential equation is

$$\frac{x}{a^2}\frac{dz}{dx} + \frac{y}{b^2}\frac{dz}{dy} = \frac{z}{c^2}.$$

Whence we find $$\frac{y^{b^2}}{z^{c^2}} = \phi\left(\frac{x^{a^2}}{z^{c^2}}\right),$$

as the equation to the trajectory.

(21) Find the curve in which the length of the arc bears a constant ratio to the intercept of the tangent on the axis of x.

If the ratio be that of m to n, we have

$$\int dx\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \frac{m}{n}\left(y\frac{dx}{dy} - x\right).$$

Whence $$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \frac{m}{n}y\frac{dy}{dx}\frac{d^2x}{dy^2},$$

$$\text{or}\quad \frac{d^2x}{(dx^2+dy^2)^{\frac{1}{2}}} = \frac{m}{n}\frac{dy}{y}.$$

Integrating, $\log\left\{\frac{dx+(dx^2+dy^2)^{\frac{1}{2}}}{dy}\right\} = \frac{m}{n}\log cy;$

therefore $$\frac{dx}{dy}+\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{\frac{1}{2}} = cy^{\frac{m}{n}}.$$

From this we get

$$2c\,dx = \left(c^2y^{\frac{m}{n}} - y^{-\frac{m}{n}}\right)dy;$$

the integral of which is

$$2cx + c' = n\left(\frac{c^2y^{\frac{m+n}{n}}}{m+n} + \frac{y^{\frac{n-m}{n}}}{m-n}\right).$$

This is the simplest case of the "curves of pursuit," and the problem may be expressed thus: A point P moves along a straight line, and is pursued by a point Q, whose velocity is to that of P always in the ratio of m to n, find the path of Q, supposing the line joining the points at the beginning of the motion not to coincide with the direction of the motion of P.

Bouguer, *Mémoires de l'Académie*, 1732.

(22) Find the curve in which the radius of curvature is equal to the normal.

The radius of curvature is $$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}}.$$

The normal is $$y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

Now the radius of curvature and the normal may lie on the same or on different sides of the curve: this will be indicated by taking the radius of curvature with a negative or a positive sign. Hence by the condition,

$$y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \mp\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}},$$

$$\text{or}\quad \frac{\frac{d^2y}{dx^2}}{1+\left(\frac{dy}{dx}\right)^2} = \mp\frac{1}{y}.$$

Multiplying by $\frac{dy}{dx}$ and integrating,

$$\log\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \log c \mp \log y,$$

$$\text{or}\quad \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \frac{c}{y} \text{ or } = \frac{y}{c},$$

whence $\frac{dy}{dx} = \frac{(c^2-y^2)^{\frac{1}{2}}}{y}$ or $= \frac{(y^2-c^2)^{\frac{1}{2}}}{c}$.

The first gives on integration

$$-(c^2-y^2)^{\frac{1}{2}} = x + a,$$

$$\text{or}\quad x^2+y^2-2ax = c^2-a^2,$$

the equation to a circle.

The latter gives

$$\log\{y+(y^2-c^2)^{\frac{1}{2}}\} = \frac{x}{c}+a,$$

or as we may write it for convenience

$$\log\frac{y+(y^2-c^2)^{\frac{1}{2}}}{c} = \frac{x+a}{c}.$$

From this it is easy to see that

$$y = \frac{c}{2}\left(\epsilon^{\frac{x+a}{c}} + \epsilon^{-\frac{x+a}{c}}\right),$$

which is the equation to the catenary.

(23) Find the curve which has an evolute similar to itself.

This remarkable problem if attempted by means of reference to rectilinear co-ordinates would be quite impracticable. Euler,* however, has by a most ingenious method reduced the problem to a very simple shape. Instead of using rectilinear or polar co-ordinates he refers the curve to its radius of curvature, and the angle which that line makes with a line passing through the first point of the curve to be investigated. There is thus nothing left arbitrary except the first point in the curve.

Let ρ be the radius of curvature, ϕ the angle which it makes with the line passing through the first point of the curve. Then if s be the arc of the curve measured from the same point,

$$ds = \rho d\phi.$$

But $$\phi = \tan^{-1} p \text{ if } p = \frac{dy}{dx};$$

and as $ds = (1 + p^2)^{\frac{1}{2}} dx$, we have

$$x = \int d\phi\, \rho \cos\phi, \quad y = \int d\phi\, \rho \sin\phi;$$

and x and y are thus known in terms of the co-ordinates which we are to employ.

Now let AS (fig. 61) be the curve, $A'S'$ its evolute which is to be similar to AS, and let $A''S''$ be the evolute of $A'S'$, and therefore similar to it and to AS: let PP', QQ' be the radii of curvature at the successive points P, Q. Then considering the small arcs PQ and $P'Q'$ as coinciding with the arcs of the circles of curvature, we see that the elemental sectors $PQ'Q$ and $P'Q''Q'$ are similar, and therefore

$$PQ : P'Q' = PP' : P'P''.$$

Putting $PQ = ds$, $P'Q' = ds'$, $PP' = \rho$, $P'P'' = \rho'$,

this gives $$\rho' ds = \rho ds'.$$

But by the property of the evolute $ds' = d\rho$; therefore

$$\rho' ds = \rho d\rho.$$

But $$ds = \rho d\phi; \text{ therefore } \rho' = \frac{d\rho}{d\phi}.$$

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* *Nova Acta Petrop.* Vol. I. p. 75.

In order that the evolute may be similar to the curve, we must have

$$\rho' = a\rho,$$

a being the coefficient of similarity.

Hence for determining the curve we have the equation

$$\frac{d\rho}{d\phi} = a\rho.$$

This being a linear equation is easily integrated, and the result is

$$\rho = C\epsilon^{a\phi}.$$

Substituting this value of ρ in the expression for x and y,

$$x = C\int d\phi\, \epsilon^{a\phi} \cos\phi, \quad y = C\int d\phi\, \epsilon^{a\phi} \sin\phi.$$

Whence

$$x + A = \frac{C\epsilon^{a\phi}}{1 + a^2}(a\cos\phi + \sin\phi) = \frac{C\epsilon^{a\phi}}{1 + a^2}\,\frac{a + p}{(1 + p^2)^{\frac{1}{2}}},$$

$$y + B = \frac{C\epsilon^{a\phi}}{1 + a^2}(a\sin\phi - \cos\phi) = \frac{C\epsilon^{a\phi}}{1 + a^2}\,\frac{ap - 1}{(1 + p^2)^{\frac{1}{2}}}.$$

From which we have

$$(y + B)(a + p) = (x + A)(ap - 1).$$

Or putting x and y for $x + A$ and $y + B$, which does not affect $p = \frac{dy}{dx}$ this becomes

$$a(xdy - ydx) = xdx + ydy,$$

which is the differential equation to the logarithmic spiral. That curve therefore is the only one which has an evolute similar to itself.

Euler in the memoir referred to above has considered the question much more generally, for he investigates the nature of the curve which has its n^{th} evolute similar to itself, as well as the curve which has an evolute similar to itself but placed in an inverse position. This last is reduced to the previous case, for if the evolute be similar to the original curve but in an inverted position, the second evolute will also be similar to the original curve and in the same position, and its radii of curvature will diminish while those of the first evolute increase, as will be seen in (fig. 62). It is easy to see that this condition is expressed symbolically by affecting the coefficient of similarity with a negative sign. The general equation for a curve which has its n^{th} evolute directly similar to itself is

$$\frac{d^n \rho}{d\phi^n} - a^n \rho = 0,$$

that which has its n^{th} evolute inversely similar is

$$\frac{d^{2n} \rho}{d\phi^{2n}} + a^{2n} \rho = 0.$$

(24) Let us investigate a particular case of this last problem when $n = 1$ and $a^2 = 1$, which implies that the evolute is *equal* to the curve but in an inverted position.

The equation then becomes

$$\frac{d^2 \rho}{d\phi^2} + \rho = 0.$$

The integral of which is

$$\rho = C \cos \phi + C_1 \sin \phi = C \cos (\phi + a).$$

Since the angle a depends only on the line from which ϕ is measured we may make it equal to zero, so that

$$\rho = C \cos \phi, \text{ then}$$

$$x = C \int d\phi \cos^2 \phi = \frac{C}{2} \int (1 + \cos 2\phi)\, d\phi = \frac{C}{4} (2\phi + \sin 2\phi),$$

making $x = 0$ when $\phi = 0$.

Also $y = C \int d\phi \sin \phi \cos \phi = -\frac{C}{4} \cos 2\phi + A.$

If $y = 0$ when $\phi = 0$, $A = \frac{C}{4}$; therefore

$$y = \frac{C}{4} (1 - \cos 2\phi).$$

Eliminating ϕ we have

$$x = \frac{C}{4} \operatorname{vers}^{-1} \frac{4y}{C} + \left(\frac{Cy}{2} - y^2\right)^{\frac{1}{2}},$$

the equation to a cycloid referred to its vertex.

If $a^2 < 1$, the curve is an epicycloid.

If $a^2 > 1$, the curve is a hypocycloid.

(25) In speaking of the cycloid I mentioned a property belonging to it which was discovered by John Bernoulli, viz., that if BC (Fig. 21) be any curve, the tangents at the extremities of which are at right angles to each other, and if this be developed, beginning from C, and if the involute CD be again developed, beginning from D, and so on in succession, the successive involutes approach continually nearer and nearer to the cycloid, and ultimately do not differ sensibly from that curve. The following demonstration of this remarkable proposition is taken from Legendre, *Exercices de Calcul Integral*, Vol. II. p. 541.

Draw the successive tangents MP, PN, NQ ... which will be alternately perpendicular and parallel to the first, from the nature of involutes. Let θ be the angle which MP makes with the line AB, and put

$$\begin{aligned} \text{arc } CM &= x, & \text{arc } CB &= a, \\ \text{arc } CP &= z, & \text{arc } CD &= b, \\ \text{arc } EN &= x', & \text{arc } ED &= a', \\ \text{arc } EQ &= z', & \text{arc } EF &= b', \end{aligned}$$

and so on in succession. Then if we were to draw tangents, making angles $= d\theta$ with the other tangents, we should have

$$d\theta = \frac{dz}{MP} = \frac{dx'}{PN} = \frac{dz'}{QN} = \dots$$

But from the nature of involutes,

$$MP = CM = x, \quad PN = DP = b - z, \quad QN = EN = x', \text{ \&c.}$$

Hence $$d\theta = \frac{dz}{x} = \frac{dx'}{b - z} = \frac{dz'}{x'} = \frac{dx''}{b' - z'} \dots$$

From the first we have $z = \int x d\theta$, which ought to vanish when $\theta = 0$, and to become equal to b when $\theta = \frac{\pi}{2}$. The second equation gives

$$dx' = b d\theta - z d\theta = b d\theta - d\theta \int x d\theta,$$

and $$x' = b\theta - \int^2 d\theta^2 x,$$

which, when $\theta = 0$, ought to vanish, and when $\theta = \frac{\pi}{2}$ to be equal to a'. The third equation gives

$$dz' = x' d\theta = b\theta d\theta - d\theta \int^2 d\theta^2 x,$$

and $$z' = \frac{b\theta^2}{1.2} - \int^3 d\theta^3 x.$$

Proceeding in this way, we have

$$x^{(n)} = b^{(n-1)}\theta - \frac{b^{(n-2)}\theta^3}{1.2.3} + \frac{b^{(n-3)}\theta^5}{1.2.3.4.5} - \&c. \pm \int^{2n} d\theta^{2n} x,$$

$$z^{(n)} = \frac{b^{(n-1)}\theta^2}{1.2} - \frac{b^{(n-2)}\theta^4}{1.2.3.4} + \&c. \qquad \pm \int^{2n+1} d\theta^{2n+1} x.$$

Now the last terms in both of these expressions continually diminish, and if n be made sufficiently large they may be neglected. This may be seen by considering that since

$$x < a, \quad \int^n d\theta^n x \text{ is less than } \int^n d\theta^n a, \text{ or } \frac{\theta^n a}{1.2\ldots n};$$

and as the greatest value of θ is $\frac{\pi}{2}$, the denominator, when n is great, far surpasses the numerator, and the term diminishes continually as n increases. Neglecting then the last term and making $\theta = \frac{\pi}{2} = a$ in the second series, we have

$$b^{(n)} = \frac{b^{(n-1)}a^2}{1.2} - \frac{b^{(n-2)}a^4}{1.2.3.4} + \&c.$$

From this it appears that when n is large the series

$$b + b'y + b''y^2 + \&c. + b^{(n)}y^n + \&c.$$

may be considered as a recurring series formed from a fraction of which the numerator is a polynomial in y of a finite number of terms, and the denominator is

$$1 - \frac{a^2 y}{1.2} + \frac{a^4 y^2}{1.2.3.4} - \&c. = \cos (a y^{\frac{1}{2}}).$$

Now $\cos (a y^{\frac{1}{2}}) = \cos \left(\frac{\pi y^{\frac{1}{2}}}{2}\right) = (1 - y) \left(1 - \frac{y}{3^2}\right) \left(1 - \frac{y}{5^2}\right) \ldots;$

and if $f(y)$ be the numerator, we may assume

$$\frac{f(y)}{\cos (a y^{\frac{1}{2}})} = \frac{N_1}{1 - y} + \frac{N_2}{1 - \frac{y}{3^2}} + \frac{N_3}{1 - \frac{y}{5^2}} + \&c.$$

by the theory of rational fractions. Now the coefficient of y^n in $\dfrac{f(y)}{\cos (a y^{\frac{1}{2}})}$ is supposed to be $b^{(n)}$ when n is large; and on the other side the coefficient of y^n is

$$N_1 + \frac{N_2}{3^{2n}} + \frac{N_3}{5^{2n}} + \&c.$$

If n be indefinitely increased, this is reduced to N_1, which is independent of n. Therefore $b^{(n)}$ is independent of n when n is very large: hence

$$b^{(n)} = b^{(n-1)} = b^{(n-2)} \ldots$$

and $$z^{(n)} = b^{(n)} \left(\frac{\theta^2}{1.2} - \frac{\theta^4}{1.2.3.4} + \&c.\right) = b^{(n)} (1 - \cos \theta).$$

Similarly $$x^{(n)} = b^{(n)} \sin \theta.$$

These equations belong to a cycloid, in which $\frac{1}{2} b^{(n)}$ is the radius of the generating circle. Thence follows the proposition.

(26) Find the surface, such that the intercept of the tangent plane on the axis of z is proportional to the distance from the origin.

The intercept of the tangent plane on the axis of z is

$$z - px - qy;$$

hence we have

$$z - px - qy = n\,(x^2 + y^2 + z^2)^{\frac{1}{2}}.$$

The integral of this equation is

$$x^{n-1}\{z + (x^2 + y^2 + z^2)^{\frac{1}{2}}\} = \phi\left(\frac{y}{x}\right).$$

(27) Find the surface in which the co-ordinates of the point where the normal meets the plane of xy are proportional to the corresponding co-ordinates of the surface.

The equations to the normal being

$$x' - x + p\,(z' - z) = 0, \quad y' - y + q\,(z' - z),$$

we have when $z' = 0$,

$$x' = x + pz, \quad y' = y + qz,$$

therefore $\quad x + pz = mx, \quad y + qz = ny.$

Substituting these values in

$$dz = p\,dx + q\,dy,$$

and integrating, we find

$$z^2 = (m - 1)\,x^2 + (n - 1)\,y^2 + C,$$

which is the equation to a surface of the second order.

(28) To find the equation to the surface at every point of which the radii of curvature are equal and of the same sign.

The conditions that this should be the case are

$$\frac{1 + p^2}{r} = \frac{pq}{s} = \frac{1 + q^2}{t};$$

$$\text{or} \quad \frac{p}{1 + p^2}\,\frac{dp}{dx} = \frac{1}{q}\,\frac{dq}{dx}, \quad \frac{q}{1 + q^2}\,\frac{dq}{dy} = \frac{1}{p}\,\frac{dp}{dy}.$$

Integrating these as ordinary equations and replacing the arbitrary constants, in the first equation by an arbitrary function (Y) of y, in the second by an arbitrary function (X) of x, we find

$$1 + p^2 = Yq^2, \quad 1 + q^2 = Xp^2.$$

From these we find

$$p = \left(\frac{1+Y}{XY-1}\right)^{\frac{1}{2}}, \quad q = \left(\frac{1+X}{XY-1}\right)^{\frac{1}{2}}.$$

But p and q ought by their nature to satisfy the equation $\frac{dp}{dy} = \frac{dq}{dx}$, which in the present case is

$$(1+X)^{-\frac{3}{2}}\frac{dX}{dx} = (1+Y)^{-\frac{3}{2}}\frac{dY}{dy}.$$

Now whatever be the form of the functions X and Y, this equation is of the form $\phi(x) = \psi(y)$, and it can therefore subsist only when each side is equal to a constant. Let this constant be represented by $\frac{2}{r}$; then

$$(1+X)^{-\frac{3}{2}}\frac{dX}{dx} = \frac{2}{r}, \quad (1+Y)^{-\frac{3}{2}}\frac{dY}{dy} = \frac{2}{r},$$

whence on integration we obtain

$$\frac{r}{(1+X)^{\frac{1}{2}}} = a - x, \quad \frac{r}{(1+Y)^{\frac{1}{2}}} = b - y,$$

a and b being arbitrary constants. If from these we take the values of X and Y and substitute them in those of p and q we have

$$p = \frac{(a-x)}{\{r^2-(a-x)^2-(b-y)^2\}^{\frac{1}{2}}}, \quad q = \frac{b-y}{\{r^2-(a-x)^2-(b-y)^2\}^{\frac{1}{2}}}.$$

Putting these values into the formula

$$dz = p\,dx + q\,dy,$$

and integrating, we have

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2,$$

which is the equation to a sphere.

Monge, *Analyse Appliquée.*

CHAPTER XI.

EVALUATION OF DEFINITE INTEGRALS.

When we are able to effect the integration of any function, the determination of its value between certain limits of the independent variable offers in general no difficulty, as we have merely to subtract its value at one limit from its value at another. There are however many functions, the Definite Integrals of which we are able to find, although the indefinite integral cannot be expressed in finite terms. The evaluation of these integrals has become one of the most important branches of the Integral Calculus, in consequence of the numerous applications which are made of them both in pure mathematics and in physics: it is to functions of this kind that the examples in the following paper refer.

The methods for evaluating those definite integrals whose general values cannot be found are very various, but they can generally be classed under the following heads.

(1) Expansion of the function into series, integration of each term separately, and summation of the result.

(2) Differentiation and integration with respect to some quantity not affected by the original sign of integration.

(3) Integration by parts of a known definite integral, so as to obtain a relation between it and an unknown one.

(4) Multiplication of several definite integrals together, so as to obtain a multiple integral, and, by a change of the variables in this, converting it into another multiple integral, coinciding with the first at the limits, and admitting of integration. By this means a relation is found between the definite integrals multiplied together, which frequently enables us to discover their values.

(5) Conversion of the function by means of impossible quantities into a form admitting of integration.

These different methods will be best understood by their application to the following examples.

We shall begin with the function known as the Second Eulerian Integral, because, though its exact value cannot be found generally, its properties have been much studied, and to it a number of other integrals are reduced.

1. *Second Eulerian Integral.*

The definite integral $\int_0^\infty dx\,\epsilon^{-x}x^{n-1}$, when n is a whole number, is easily seen by the method of reduction in Ex. (13), Chap. II. of the *Integ. Calc.* to be

$$n(n-1)\dots 3.2.1.$$

When, however, n is a fraction, its value can be found only in certain cases, but it possesses many remarkable properties which render it of the greatest importance in the Theory of Definite Integrals. It was first studied by Euler, who seems at an early period to have seen its importance, and has devoted several memoirs to the investigation of its properties; on this account Legendre has named it after him, at once for the purposes of characterizing the function and honouring that great mathematician. To distinguish it from another integral with which also Euler had much occupied himself, and of which we shall afterwards treat, it is usually called the "Second Eulerian Integral," and Legendre has affixed to it the characteristic symbol Γ, applied to the index, so that he writes

$$\int_0^\infty dx\,\epsilon^{-x}x^{n-1} = \Gamma(n),$$

which notation we shall adopt. Throughout the following investigations n is supposed to be greater than 0.

In the first place we remark that by a change of the independent variable this integral may be put under other forms which are sometimes more convenient in practice than that which we have used.

Thus if we put $\epsilon^{-x} = y$, the corresponding limits are

$$x = 0, \quad y = 1; \quad x = \infty, \quad y = 0,$$

and the integral takes the form

$$(a) \qquad \Gamma(n) = \int_0^1 dx \left(\log \frac{1}{x}\right)^{n-1}.$$

This is the shape under which the integral has been usually treated both by Euler and Lagrange, but it is scarcely so convenient as the preceding.

Again, if we put $x^n = z$, the limits remain the same as before, and we have

$$(b) \qquad \Gamma(n) = \frac{1}{n} \int_0^\infty dz\, \epsilon^{-z^{\frac{1}{n}}}.$$

This last form is the most convenient for determining the value of the integral in one remarkable case when it can be found in finite terms. If $n = \frac{1}{2}$

$$\int_0^\infty dx\, \epsilon^{-x} x^{-\frac{1}{2}} = \Gamma\left(\tfrac{1}{2}\right) = 2 \int_0^\infty dz\, \epsilon^{-z^2}.$$

Let $k = \int_0^\infty dz\, \epsilon^{-z^2}$:

then as the value of the definite integral is independent of the variable, we have also

$$k = \int_0^\infty dy\, \epsilon^{-y^2},$$

and therefore multiplying these together,

$$k^2 = \int_0^\infty dz \epsilon^{-z^2} . \int_0^\infty dy\, \epsilon^{-y^2} = \int_0^\infty \int_0^\infty dy\, dz\, \epsilon^{-(y^2+z^2)};$$

since y and z are independent. Now assume

$$z = r \cos\theta, \qquad y = r \sin\theta, \text{ then}$$

$$z^2 + y^2 = r^2, \qquad dy\, dz = r\, dr\, d\theta.$$

To determine the limits we observe that y and z never become negative, and therefore θ must vary from 0 to $\dfrac{\pi}{2}$, while r varies from 0 to ∞, so that we have

$$k^2 = \int_0^\infty \int_0^{\frac{1}{2}\pi} dr\, d\theta\, r \epsilon^{-r^2} = \tfrac{1}{4}\pi; \quad \text{whence}$$

$$(c) \qquad k = \tfrac{1}{2}\pi^{\frac{1}{2}} \text{ and } \Gamma\left(\tfrac{1}{2}\right) = \pi^{\frac{1}{2}}.$$

We shall now demonstrate the more important properties of the function $\Gamma(n)$ referring the reader who wishes, for a more detailed exposition of them to Legendre, *Exercices de Calcul Intégral*, Tom. I. and II.

If we integrate by parts the expression $\int dx\, \epsilon^{-x} x^n$ we have

$$\int dx\, \epsilon^{-x} x^n = -\epsilon^{-x} x^n + n \int dx\, \epsilon^{-n} x^{n-1}.$$

The integrated part vanishes at both limits, so that

$$(d) \qquad \Gamma(n+1) = n\Gamma(n).$$

This may be looked on as a characteristic property of the function Γ, and is of the greatest importance, as by means of it we can reduce the calculation of $\Gamma(n)$ from the case when $n > 1$ to that when it is < 1, and we have therefore to occupy ourselves only with the values of n which lie between 0 and 1.

If n be a proper fraction,

$$\Gamma(n) = \int_0^\infty dx\, \epsilon^{-x} x^{n-1}; \qquad \Gamma(1-n) = \int_0^\infty dy\, \epsilon^{-y} y^{-n};$$

and therefore

$$\Gamma(n)\,\Gamma(1-n) = \int_0^\infty dx\, \epsilon^{-x} x^{n-1} \int_0^\infty dy\, \epsilon^{-y} y^{-n}$$
$$= \int_0^\infty \int_0^\infty dx\, dy\, \epsilon^{-(x+y)} x^{n-1} y^{-n}.$$

To reduce this, we shall use the transformation of *Jacobi*, given in Chap. III. Sec. 2, Ex. (7) of the *Diff. Calc.*

Assume $x + y = u$, $y = uv$, so that $dx\,dy = u\,du\,dv$: the limits of u and v corresponding to those of x and y, are $u = 0$, $u = \infty$, $v = 0$, $v = 1$; therefore

$$\Gamma(n)\,\Gamma(1-n) = \int_0^\infty \int_0^1 du\, dv\, \epsilon^{-u} v^{-n} (1-v)^{n-1};$$

or, integrating with respect to u between its limits,

$$\Gamma(n)\,\Gamma(1-n) = \int_0^1 dv\, v^{-n} (1-v)^{n-1}.$$

To find the value of this integral, assume $v = (\sin\theta)^2$; then, as to the limits $x = 0$, $x = 1$, correspond $\theta = 0$, $\theta = \frac{1}{2}\pi$, we have $\qquad \Gamma(n)\,\Gamma(1-n) = 2\int_0^{\frac{1}{2}\pi} d\theta\, (\tan\theta)^{1-2n}$.

Now $\tan\theta = (-)^{-\frac{1}{2}} \dfrac{1 - \epsilon^{-(-)^{\frac{1}{2}} 2\theta}}{1 + \epsilon^{-(-)^{\frac{1}{2}} 2\theta}}$, and it is therefore obvious that $(\tan\theta)^{1-2n}$ may be expanded into a series of the form

$$(-)^{\frac{2n-1}{2}}\{1 + A_1\epsilon^{-(-)^{\frac{1}{2}}2\theta} + A_2\epsilon^{-(-)^{\frac{1}{2}}4\theta} + \&c.\}$$

$$= (-)^{\frac{2n-1}{2}}\left\{\begin{array}{l}1 + A_1\cos 2\theta + A_2\cos 4\theta + \&c. \\ -(-)^{\frac{1}{2}}(A_1\sin 2\theta + A_2\sin 4\theta + \&c.\end{array}\right\}.$$

Also $$(-)^{\frac{1-2n}{2}} = \cos(1-2n)\frac{\pi}{2} + (-)^{\frac{1}{2}}\sin(1-2n)\frac{\pi}{2}.$$

$$= \sin n\pi + (-)^{\frac{1}{2}}\cos n\pi.$$

Substituting the series for $(\tan\theta)^{1-2n}$, multiplying by $\sin n\pi + (-)^{\frac{1}{2}}\cos n\pi$, and equating real parts, we have

$$\sin n\pi\,\Gamma(n)\,\Gamma(1-n) = 2\int_0^{\frac{\pi}{2}} d\theta\,(1 + A_1\cos 2\theta + A_2\cos 4\theta + \&c.)$$
$$= \pi;$$

since the periodic terms vanish at both limits. Hence

$$(e) \qquad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\,\text{*}.$$

It is easily seen that the value of $\Gamma(\frac{1}{2})$ is found at once from this equation, by putting $n = \frac{1}{2}$; and generally, if we know the value of $\Gamma(n)$ from $n = 1$ to $n = \frac{1}{2}$, we know its value from 0 to $\frac{1}{2}$.

From the preceding theorem a more general one may be derived. Let n be a positive integer, then will

$$(f) \quad \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}};$$

1st. Let n be even: Then there are $\frac{1}{2}n - 1$ pairs of factors of the form $\Gamma(r)\,\Gamma(1-r)$, and a middle factor which is $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$. Therefore by the preceding theorem the product is equal to

$$\frac{\pi^{\frac{n-1}{2}}}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\ldots\sin\frac{(n-2)\pi}{2n}}.$$

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* This demonstration of a Theorem discovered by Euler is given by Mr Greatheed in the *Camb. Math. Journal*, Vol. I. p. 17.

Now by a known theorem, when n is even, we have

$$\frac{\sin nz}{\sin z} = 2^{n-1}\sin\left(\frac{\pi}{n} - z\right)\sin\left(\frac{\pi}{n} + z\right)\dots\sin\left(\frac{\pi}{2} - z\right);$$

which, when $z = 0$, gives

$$n2^{-(n-1)} = \left(\sin\frac{\pi}{n}\right)^2\left(\sin\frac{2\pi}{n}\right)^2\dots\left\{\sin\frac{(n-2)\pi}{2n}\right\}^2:$$

so that we find

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

2nd. When n is odd, there is no middle factor, and the number of double factors is $\frac{1}{2}(n-1)$. Also in this case we have

$$\frac{\sin nz}{\sin z} = 2^{n-1}\sin\left(\frac{\pi}{n} - z\right)\sin\left(\frac{\pi}{n} + z\right)\dots$$
$$\sin\left(\frac{n-1}{2n}\pi - z\right)\sin\left(\frac{n-1}{2n}\pi + z\right);$$

and by making $z = 0$, we have as before

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}}n^{-\frac{1}{2}}.$$

A still more general theorem is the following:

$$(g)\quad \Gamma(x)\Gamma\left(x+\frac{1}{n}\right)\dots\Gamma\left(x+\frac{n-1}{n}\right) = \Gamma(nx)(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nx};$$

but for the demonstration the reader is referred to Legendre's work.

To this definite integral $\Gamma(n)$ others may be easily reduced.

Thus if we have the integral

$$\int_0^1 dx\, x^{m-1}\left(\log\frac{1}{x}\right)^{n-1},$$

by assuming $x^m = z$, it becomes

$$\frac{1}{m^n}\int_0^1 dz\left(\log\frac{1}{z}\right)^{n-1},$$

so that

$$(h)\quad \int_0^1 dx\, x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} = \frac{1}{m^n}\Gamma(n).$$

Again, since by Ex. (1, *c*) we have

$$\int_0^\infty dx\, \epsilon^{-ax^2} = \frac{\pi^{\frac{1}{2}}}{2}\frac{1}{a^{\frac{1}{2}}},$$

on differentiating $2n$ times with respect to a and then making $a = 1$, we obtain

$$(k)\quad \int_0^\infty dx\, x^{2n}\epsilon^{-ax^2} = \frac{1.3.5\ldots(2n-1)}{2^n}\frac{\pi^{\frac{1}{2}}}{2}.$$

(2) The first Eulerian integral is

$$\int_0^1 dx \frac{x^{p-1}}{(1-x^n)^{\frac{n-q}{n}}} = \left(\frac{p}{q}\right),$$

according to the notation adopted by Euler, and, after him, by Legendre; the value of the integral being supposed to change in consequence of the variation of p and q, n remaining constant. This form of the integral however is not the most convenient in practice, and we shall use another, formed from the present by putting $x^n = y$, when it becomes

$$\frac{1}{n}\int_0^1 dy\, y^{\frac{p}{n}-1}(1-y)^{\frac{q}{n}-1}.$$

Putting $\frac{p}{n} = l$, $\frac{q}{n} = m$, we shall designate the definite integral by a symbol of functionality applied to these letters as the symbol Γ is used for the second Eulerian integral: the letter we shall use is the digamma $\digamma$, so that we write

$$(a)\qquad \int_0^1 dx\, x^{l-1}(1-x)^{m-1} = \digamma(l, m).$$

The most important properties of this integral are those by which it is connected with the second Eulerian integral founded on the theorem

$$(b)\qquad \digamma(l, m) = \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m)}.$$

To demonstrate this we shall proceed as in Ex. (1, e), which is a particular case of this theorem;

$$\Gamma(l) \cdot \Gamma(m) = \int_0^\infty \int_0^\infty dx\, dy\, \epsilon^{-(x+y)} x^{l-1} y^{m-1};$$

and on transformation

$$\Gamma(l) \cdot \Gamma(m) = \int_0^\infty du\, \epsilon^{-u} u^{l+m-1} \int_0^1 dv\, v^{l-1} (1-v)^{m-1}.$$

Now

$$\int_0^\infty du\, \epsilon^{-u} u^{l+m-1} = \Gamma(l+m) \text{ and } \int_0^1 dv\, v^{l-1} (1-v)^{m-1} = \mathsf{F}(l, m);$$

$$\text{therefore} \quad \mathsf{F}(l, m) = \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l+m)}.$$

As l and m enter symmetrically into the second side of the equation, it follows that

$$(c) \qquad \mathsf{F}(l, m) = \mathsf{F}(m, l).$$

Also since

$$\mathsf{F}(l, m) = \frac{\Gamma(l)\, \Gamma(m)}{\Gamma(l+m)}, \quad \text{and } \mathsf{F}(l+m, n) = \frac{\Gamma(l+m)\, \Gamma(n)}{\Gamma(l+m+n)};$$

by multiplying these together we have

$$\mathsf{F}(l, m) \cdot \mathsf{F}(l+m, n) = \frac{\Gamma(l)\, \Gamma(m)\, \Gamma(n)}{\Gamma(l+m+n)};$$

and as the second side is a symmetrical function of l, m, n, it follows that these letters may be interchanged, so that

$$(d) \quad \mathsf{F}(l,m) \cdot \mathsf{F}(l+m,n) = \mathsf{F}(l,n) \cdot \mathsf{F}(l+n,m) = \mathsf{F}(m,n) \cdot \mathsf{F}(m+n,l).$$

The integral $\int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}}$ may be reduced to the first Eulerian integral by assuming

$$\frac{x}{x+a} = \frac{y}{1+a},$$

when it becomes

$$\frac{1}{a^\beta (1+a)^\alpha} \int_0^1 dy\, y^{\alpha-1} (1-y)^{\beta-1},$$

the limits of y being the same as those of x. Hence

$$(e) \quad \int_0^1 dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{(x+a)^{\alpha+\beta}} = \frac{1}{a^\beta (1+a)^\alpha} \frac{\Gamma(\alpha)\, \Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

Abel, *Œuvres*, Vol. I. p. 95.

(3) The property (*b*) of the first Eulerian integral may be extended to a large class of multiple integrals by the following theorem due to M. Lejeune Dirichlet*.

(*a*) Let $V = \int dx \int dy \int dz \dots x^{a-1} y^{b-1} z^{c-1} \dots$ in which the limiting values of $x, y, z \dots$ are given by the condition

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \&c. \underset{<}{=} 1,$$

$a, b, c \dots \alpha, \beta, \gamma \dots p, q, r \dots$ being positive quantities;

then will $$V = \frac{\alpha^a \beta^b \gamma^c \dots}{pqr \dots} \frac{\Gamma\left(\frac{a}{p}\right) \Gamma\left(\frac{b}{q}\right) \Gamma\left(\frac{c}{r}\right) \dots}{\Gamma\left(1 + \frac{a}{p} + \frac{b}{q} + \frac{c}{r} + \dots\right)}.$$

The equation of the limits may be made linear by putting x for $\left(\frac{x}{\alpha}\right)^p$, y for $\left(\frac{y}{\beta}\right)^q$, z for $\left(\frac{z}{\gamma}\right)^r$, &c., in which case the integral becomes

$$V = \frac{\alpha^a \beta^b \gamma^c \dots}{pqr \dots} \int dx \int dy \int dz \dots x^{l-1} y^{m-1} z^{n-1} \dots,$$

with the condition

$$x + y + z + \dots \underset{<}{=} 1;$$

where $$l = \frac{a}{p}, \quad m = \frac{b}{q}, \quad n = \frac{c}{r}, \ \&c.$$

Hence if we know the integral

$$U = \int dx \int dy \int dz \dots x^{l-1} y^{m-1} z^{n-1} \dots,$$

with the previous condition for determining the limits, we can find V.

When the variables are two in number, it is easy to see that the integral is identical with that called the first Eulerian. Let us suppose therefore that there are three variables. Then

$$U = \int_0^1 dx\, x^{l-1} \int_0^{y_1} dy\, y^{m-1} \int_0^{z_1} dz\, z^{n-1},$$

where $$y_1 = 1 - x, \quad z_1 = 1 - x - y.$$

* Liouville's *Journal*, Vol. IV. p. 168.

Assume $z = vz_1$, $y = uy_1$, the limits of u and v are then 0 and 1, and U takes the form

$$U = \int_0^1 dx\, x^{l-1} \int_0^1 du\, u^{m-1} y_1^m \int_0^1 dv\, v^{n-1} z_1^n.$$

But as $y_1 = 1 - x$, and $z_1 = y_1 - uy_1 = (1-x)(1-u)$, the integral becomes

$$U = \int_0^1 dx\, x^{l-1}(1-x)^{m+n}. \int_0^1 du\, u^{m-1}(1-u)^n. \int_0^1 dv\, v^{n-1}.$$

The integrations with respect to the different variables may now be effected separately, and we have

$$\int_0^1 dv\, v^{n-1} = \frac{\Gamma(n)}{\Gamma(n+1)}, \quad \int_0^1 du\, u^{m-1}(1-u)^n = \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)},$$

$$\int_0^1 dx\, x^{l-1}(1-x)^{m+n} = \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)},$$

so that
$$U = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(1+l+m+n)}.$$

In like manner we might find the value of U when there are four variables, and so on for any number; and hence, also, the value of V, as stated, is deduced.

(*b*) By a similar process M. Liouville has proved the still more general theorem, that if

$$W = \int dx \int dy \int dz \ldots x^{a-1} y^{b-1} z^{c-1} \ldots f\left\{\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots\right\},$$

where the limits are given by the condition

$$\left(\frac{x}{\alpha}\right)^p + \left(\frac{y}{\beta}\right)^q + \left(\frac{z}{\gamma}\right)^r + \ldots \leqq h;$$

then will

$$W = \frac{\alpha^a \beta^b \gamma^c \ldots}{pqr\ldots} \cdot \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)\ldots}{\Gamma(l+m+n+\ldots)} \cdot \int_0^h du\, f(u)\, u^{l+m+n-1}.$$

Liouville's *Journal*, Vol. IV. p. 231.

(*c*) As an example of this last formula, take

$$W = \int dx \int dy \int dz \frac{1}{(1-x^2-y^2-z^2)^{\frac{1}{2}}},$$

the limits being given by the condition

$$x^2 + y^2 + z^2 \stackrel{=}{<} 1.$$

In this case we find

$$W = \frac{1}{8} \frac{\{\Gamma(\frac{1}{2})\}^3}{\Gamma(\frac{3}{2})} \int_0^1 \frac{v^{\frac{1}{2}} dv}{(1-v)^{\frac{1}{2}}}.$$

Now $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\pi^{\frac{1}{2}}$, and

$$\int_0^1 \frac{v^{\frac{1}{2}} dv}{(1-v)^{\frac{1}{2}}} = 2 \int_0^1 \frac{x^2\, dx}{(1-x^2)^{\frac{1}{2}}} = \frac{\pi}{2};$$

$$\text{therefore } W = \frac{\pi^2}{8}.$$

Generally we have, the number of variables being n,

$$\iiint \dots \frac{dx\, dy\, dz \dots}{(1 - x^2 - y^2 - z^2 - \dots)^{\frac{1}{2}}} = \frac{\pi^{\frac{1}{2}(n+1)}}{2^n \Gamma\{\frac{1}{2}(n+1)\}}.$$

(d) Again, if $W = \int dx \int dy \left(\frac{1 - x^2 - y^2}{1 + x^2 + y^2}\right)^{\frac{1}{2}}$,

the limiting values being given by the condition

$$x^2 + y^2 \stackrel{=}{<} 1.$$

It will be easily seen that

$$W = \frac{1}{4} \frac{\{\Gamma(\frac{1}{2})\}^2}{\Gamma(1)} \int_0^1 dv \left(\frac{1-v}{1+v}\right)^{\frac{1}{2}},$$

whence $$W = \frac{\pi}{4}\left(\frac{\pi}{2} - 1\right).$$

(4) M. Catalan* has shewn how to evaluate a definite multiple integral which depends on those which have just been considered. It is

$$V = \iint \dots \frac{dx_1\, dx_2 \dots dx_{n-1}}{x_n} f(a_1 x_1 + a_2 x_2 + \dots + a_n x_n), \quad (1),$$

in which $x_n^2 = 1 - x_1^2 - x_2^2 - \dots - x_{n-1}^2$, and the limiting values of the $n-1$ independent variables are given by the condition

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* Liouville's *Journal*, Vol. VI. p. 81.

$$x_1^2 + x_2^2 + \dots + x_{n-1}^2 \overline{\overline{<}} 1.$$

Assume
$$\left.\begin{aligned} x_1 &= p_1 u_1 + q_1 u_2 + \dots + r_1 u_n \\ x_2 &= p_2 u_1 + q_2 u_2 + \dots + r_2 u_n \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ x_n &= p_n u_1 + q_n u_2 + \dots + r_n u_n \end{aligned}\right\} \quad (2),$$

where the coefficients of u_1 are

$$p_1 = \frac{a_1}{(a_1^2 + a_2^2 \dots + a_n^2)^{\frac{1}{2}}}, \quad p_2 = \frac{a_2}{(a_1^2 + a_2^2 + \dots + a_n^2)^{\frac{1}{2}}} \text{ \&c.};$$

and the other coefficients are subject to the conditions,

$$\left.\begin{aligned} q_1^2 + q_2^2 + \dots + q_n^2 &= 1 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ r_1^2 + r_2^2 + \dots + r_n^2 &= 1 \\ p_1 q_1 + p_2 q_2 + \dots + p_n q_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p_1 r_1 + p_2 r_2 + \dots + p_n r_n &= 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ q_1 r_1 + q_2 r_2 + \dots + q_n r_n &= 0 \end{aligned}\right\} \quad (3).$$

The number of constants in (2) is $n(n-1)$, the number of conditions in (3) is $\frac{1}{2}n(n+1)$, so that there are $\frac{1}{2}n(n-3)$ arbitrary coefficients.

Adding the squares of (2) we have by the conditions (3)

$$u_1^2 + u_2^2 + \dots + u_n^2 = x_1^2 + x_2^2 + \dots + x_n^2 = 1.$$

From the same equations we find

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = u_1 (a_1^2 + a_2^2 + \dots + a_n^2)^{\frac{1}{2}} = A u_1,$$

suppose.

Also on changing the differentials by the method given in Chap. III. Sect. 2 of the Diff. Calc. we have

$$\frac{dx_1\, dx_2 \ldots dx_{n-1}}{x_n} = \frac{du_1\, du_2 \ldots du_{n-1}}{u_n};$$

so that the integral becomes

$$V = \iint \ldots \frac{du_1\, du_2 \ldots du_{n-1}}{(1 - u_1^2 - u_2^2 \ldots u_{n-1}^2)^{\frac{1}{2}}} f(Au_1).$$

If now we integrate with respect to all the variables except u_1, the limits being given by the condition

$$u_2^2 + u_3^2 + \ldots + u_{n-1}^2 \overset{=}{<} 1 - u_1^2,$$

we have by (3, c)

$$\iint \ldots \frac{d_1 u\, d_2 u \ldots d u_{n-1}}{(1 - u_1^2 - u_2^2 \ldots u_{n-1}^2)^{\frac{1}{2}}} = \frac{\pi^{\frac{1}{2}(n-1)}}{2^{n-2}\, \Gamma\left\{\frac{1}{2}(n-1)\right\}} (1 - u_1^2)^{\frac{1}{2}(n-3)}.$$

Hence, substituting this value and integrating with respect to u_1 from 0 to 1, we have

$$(a) \quad V = \frac{\pi^{\frac{1}{2}(n-1)}}{2^{n-2}\, \Gamma\left\{\frac{1}{2}(n-1)\right\}} \int_0^1 du_1 (1 - u_1^2)^{\frac{1}{2}(n-3)} f(Au_1).$$

If we put $u_1 = \cos\theta$, this becomes

$$(b) \quad V = \frac{\pi^{\frac{1}{2}(n-1)}}{2^{n-2}\, \Gamma\left\{\frac{1}{2}(n-1)\right\}} \int_0^{\frac{\pi}{2}} d\theta\, (\sin\theta)^{n-2} f(A\cos\theta).$$

If $n = 3$, this gives

$$(c) \quad \iint \frac{dx_1\, dx_2}{(1 - x_1^2 - x_2^2)^{\frac{1}{2}}} f(a_1 x_1 + a_2 x_2 + a_3 x_3)$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} d\theta \sin\theta\, f(A\cos\theta);$$

or if $x_1 = \cos u$, $x_2 = \sin u \cos v$, $x_3 = \sin u \sin v$, and if we take the limits from $u = 0$ to $u = \pi$, and from $v = 0$ to $v = 2\pi$, it takes the form

$$(d) \quad \int_0^{\pi} \int_0^{2\pi} du\, dv \sin u\, f(a_1 \cos u + a_2 \sin u \cos v + a_3 \sin u \sin v)$$

$$= 2\pi \int_0^{\pi} d\theta \sin\theta\, f(A\cos\theta).$$

The formula under this shape was first given by Poisson in the *Mémoires de l'Institut*, Tom. III. p. 126.

(5) In the equation

$$\int_0^1 dx\, x^{r-a-1}(1-x)^{a-1} = \frac{\Gamma(a)\,\Gamma(r-a)}{\Gamma(r)},$$

if we put $1 - x = xz$, the corresponding limits are

$$x = 0, \quad z = \infty; \qquad x = 1, \quad z = 0;$$

and the transformed equation is

$$(a) \qquad \int_0^\infty dz\, \frac{z^{a-1}}{(1+z)^r} = \frac{\Gamma(a)\,\Gamma(r-a)}{\Gamma(r)}.$$

The only restriction on the generality of this result is that a must be less than r. If $r = 1$, we have

$$(b) \qquad \int_0^\infty dz\, \frac{z^{a-1}}{1+z} = \frac{\pi}{\sin a\pi}.$$

This last integral may be considered as being made up of two parts, one from $z = 0$ to $z = 1$, the other from $z = 1$ to $z = \infty$. This second part may be reduced to the same limits as the first by assuming $z = \frac{1}{z}$, when it becomes

$$\int_0^1 dz\, \frac{z^{-a}}{1+z}.$$

Hence, adding the two parts,

$$(c) \qquad \int_0^1 dz\, \frac{z^{a-1} + z^{-a}}{1+z} = \frac{\pi}{\sin a\pi}.$$

In the formula (b) put $z = y^2$ and $b = 2a$, when we find

$$(d) \qquad \int_0^\infty dy\, \frac{y^{b-1}}{1+y^2} = \frac{\pi}{2\sin\left(\frac{1}{2}b\pi\right)}.$$

(6) The integral $\int_0^\infty dx\, \frac{x^{a-1}}{1-x}$ may be considered as made up of two parts, one from $x = 0$ to $x = 1$, the other from $x = 1$ to $x = \infty$. If we put $\frac{1}{x}$ for x, the latter part becomes

$$-\int_0^1 dx\, \frac{x^{-a}}{1-x},$$

so that

$$\int_0^\infty dx \, \frac{x^{a-1}}{1-x} = \int_0^1 dx \, \frac{x^{a-1}}{1-x} - \int_0^1 dx \, \frac{x^{-a}}{1-x} = \int_0^1 dx \, \frac{x^{a-1} - x^{-a}}{1-x}.$$

Expanding the denominator and integrating from $x = 0$ to $x = 1$, we obtain the series

$$\frac{1}{a} - \frac{1}{1-a} + \frac{1}{1+a} - \frac{1}{2-a} + \frac{1}{2+a} - \&c.$$

Now by a known theorem,

$$\sin x = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\ldots$$

Taking the logarithmic differential on both sides,

$$\cot x = \frac{1}{x} - \frac{1}{\pi - x} + \frac{1}{\pi + x} - \frac{1}{2\pi - x} + \frac{1}{2\pi + x},$$

and putting $x = \pi a$, we see that

$$(a) \qquad \int_0^1 dx \, \frac{x^{a-1} - x^{-a}}{1-x} = \pi \cot a\pi = \int_0^\infty dx \, \frac{x^{a-1}}{1-x}.$$

If we put x^n for x, and $\frac{a}{n}$ for a, we have

$$(b) \qquad \int_0^\infty dx \, \frac{x^{a-1}}{1-x^n} = \frac{\pi}{n} \cot \frac{a\pi}{n}.$$

(7) The value of the integral

$$\int_0^1 dx \, \frac{x^{n-1} - 1}{\log x},$$

which is the difference of two infinite quantities, is easily found. We have

$$\int_0^1 x^{n-1} \, dx = \frac{1}{n}.$$

Integrating with respect to n and determining the constant so that the integral shall vanish when $n = 1$, there results

$$(a) \qquad \int_0^1 dx \, \frac{x^{n-1} - 1}{\log x} = \log n.$$

In like manner we find

$$(b)\quad \int_0^1 dx\,\frac{x^{m-1}-x^{n-1}}{\log x} = \log m - \log n = \log\left(\frac{m}{n}\right).$$

If we multiply

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \&c.$$

by $\dfrac{x^{m-1}-x^{n-1}}{\log x}$, and integrate from $x=0$ to $x=1$, we find

$$\int_0^1 dx\,\frac{x^{m-1}-x^{n-1}}{(1+x)\log x} = \log\frac{m}{n} - \log\frac{m+1}{n+1} + \log\frac{m+2}{n+2} - \&c.$$

$$= \log\frac{m(n+1)(m+2)(n+3)\ldots}{n(m+1)(n+2)(m+3)\ldots};$$

and if $n = 1 - m$,

$$\int_0^1 dx\,\frac{x^{m-1}-x^{-m}}{(1+x)\log x} = \log\frac{m(2-m)(2+m)(4-m)(4+m)\ldots}{(1-m)(1+m)(3-m)(3+m)\ldots}.$$

Now by the formulæ expressing the sine and cosine of an angle in products of factors, we have

$$\tan z = \frac{z\left(1-\dfrac{z}{\pi}\right)\left(1+\dfrac{z}{\pi}\right)\left(1-\dfrac{z}{2\pi}\right)\left(1+\dfrac{z}{2\pi}\right)\ldots}{\left(1-\dfrac{2z}{\pi}\right)\left(1+\dfrac{2z}{\pi}\right)\left(1-\dfrac{2z}{3\pi}\right)\left(1+\dfrac{2z}{3\pi}\right)\ldots}.$$

In this putting $z = \dfrac{\pi m}{2}$, and observing that by Wallis's theorem

$$\frac{\pi}{2} = 2\left(\frac{2}{1}\right)^2\left(\frac{4}{3}\right)^2\left(\frac{6}{5}\right)^2\left(\frac{8}{7}\right)^2\ldots\ldots$$

we see that

$$(c)\quad \int_0^1 dx\,\frac{x^{m-1}-x^{-m}}{(1+x)\log x} = \log\tan\left(m\frac{\pi}{2}\right).$$

Kummer in Crelle's *Journal*, Vol. XVII. p. 224.

(8) By integration by parts it is found that

$$\int dx\,\epsilon^{-ax}\cos rx = -\epsilon^{-ax}\frac{a\cos rx - r\sin rx}{a^2+r^2},$$

and $\int dx\, \epsilon^{-ax} \sin rx = -\epsilon^{-ax}\frac{a \sin rx + r \cos rx}{a^2 + r^2}$.

Hence taking the integrals between 0 and ∞ we have

(a) $\int_0^\infty dx\, \epsilon^{-ax} \cos rx = \frac{a}{a^2 + r^2}$, (b) $\int_0^\infty dx\, \epsilon^{-ax} \sin rx = \frac{r}{a^2 + r^2}$.

If we differentiate these expressions $(n - 1)$ times with respect to a we have by Ex. (20) and (21) of Chap. II. Sect. 1 of the Diff. Calc.

(c) $$\int_0^\infty dx\, x^{n-1} \epsilon^{-ax} \cos rx = 1.2.3 \ldots (n-1) \frac{\cos n\theta}{(a^2 + r^2)^{\frac{n}{2}}},$$

(d) $$\int_0^\infty dx\, x^{n-1} \epsilon^{-ax} \sin rx = 1.2.3 \ldots (n-1) \frac{\sin n\theta}{(a^2 + r^2)^{\frac{n}{2}}},$$

where $\theta = \tan^{-1}\frac{r}{a}$.

In these expressions n must be a positive integer; but if it be a positive fraction, the only difference is that instead of the continued product $1.2.3 \ldots (n-1)$ we must substitute the definite integral $\Gamma(n)$.

If we integrate (a) with respect to r, we have

(e) $$\int_0^\infty \frac{dx}{x} \epsilon^{-ax} \sin rx = \tan^{-1}\left(\frac{r}{a}\right),$$

no constant being added, as the integral vanishes when $r = 0$.

In this formula if we make $a = 0$, we have

(f) $$\int_0^\infty \frac{dx}{x} \sin rx = \frac{\pi}{2}.$$

If we make $a = 0$ in the formulæ (a) and (b) we have

(g) $\int_0^\infty dx \cos rx = 0$, (h) $\int_0^\infty dx \sin rx = \frac{1}{r}$.

From the integral (f) it is easy to see that

(k) $$\int_0^\infty \frac{dx}{x} \sin x \cos rx = \frac{\pi}{2},$$

when r lies between -1 and $+1$, but that it vanishes for all other values of r.

The results (g) and (h) are very remarkable as giving the real values of what are apparently indeterminate quantities, the sines and cosines of an infinite angle. For as

$$\int_0^\infty dx \cos rx = \frac{1}{r}(\sin \infty - \sin 0) = 0 \text{ by } (g),$$

it follows that $\sin \infty = 0$,

and as $\int_0^\infty dx \sin rx = -\frac{1}{r}(\cos \infty - \cos 0) = \frac{1}{r}$ by (h),

it follows that $\cos \infty = 0$;

so that both the sine and the cosine of an infinite angle are equal to zero.

In the formulæ (c) and (d) if we make $a = 0$, we find the two remarkable integrals

$$(l) \quad \int_0^\infty dx\, x^{n-1} \cos rx = \frac{1.2 \ldots (n-1)}{r^n} \cos n \frac{\pi}{2},$$

$$(m) \quad \int_0^\infty dx\, x^{n-1} \sin rx = \frac{1.2 \ldots (n-1)}{r^n} \sin n \frac{\pi}{2}.$$

If the index n lie between 0 and 1 the corresponding formulæ may be deduced without the consideration of limits involved in making $a = 0$. Since

$$\int_0^\infty da\, a^{-n} \epsilon^{-ax} = \Gamma(1-n)\, x^{n-1},$$

on multiplying both sides of this equation by $\cos rx\, dx$ and integrating from 0 to ∞, we have

$$\int_0^\infty dx \cos rx \int_0^\infty da\, a^{-n} \epsilon^{-ax} = \Gamma(1-n) \int_0^\infty dx\, x^{n-1} \cos rx.$$

But

$$\int_0^\infty dx \cos rx \int_0^\infty da\, a^{-n} \epsilon^{-ax} = \int_0^\infty da\, a^{-n} \int_0^\infty dx\, \epsilon^{-ax} \cos rx$$
$$= \int_0^\infty da \frac{a^{1-n}}{a^2 + r^2}.$$

Hence $\int_0^\infty dx\, x^{n-1} \cos rx = \frac{1}{\Gamma(1-n)} \int_0^\infty da \frac{a^{1-n}}{a^2 + r^2}.$

By the formula (d) in Ex. (5), we have

$$\int_0^\infty da\,\frac{a^{1-n}}{a^2+r^2} = \frac{1}{r^n}\,\frac{\pi}{2\sin\frac{1}{2}(2-n)\,\pi} = \frac{\pi}{2r^n\sin\left(\frac{1}{2}n\pi\right)}.$$

Therefore

$$\int_0^\infty dx\,x^{n-1}\cos rx = \frac{1}{r^n\Gamma(1-n)}\,\frac{\pi}{\sin\left(\frac{1}{2}n\pi\right)} = \frac{\Gamma(n)}{r^n}\cos\left(n\frac{\pi}{2}\right),$$

$$\text{as}\quad \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}.$$

In like manner we should find

$$\int_0^\infty dx\,x^{n-1}\sin rx = \frac{\Gamma(n)}{r^n}\sin\left(n\frac{\pi}{2}\right).$$

Thus if $n=\frac{1}{2}$, we have, since $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$, and

$$\sin\tfrac{1}{4}\pi = \cos\tfrac{1}{4}\pi = 2^{-\frac{1}{2}},$$

$$(n)\quad \int_0^\infty \frac{dx}{x^{\frac{1}{2}}}\cos rx = \left(\frac{\pi}{2r}\right)^{\frac{1}{2}} = \int_0^\infty \frac{dx}{x^{\frac{1}{2}}}\sin rx.$$

To these integrals may be reduced

$$\int_0^\infty dx\,x^{n-2}\epsilon^{-ax}\sin rx \text{ when } n<1.$$

For, on integration by parts, the integrated term vanishes at both limits, and we have

$$\int_0^\infty dx\,x^{n-2}\epsilon^{-ax}\sin rx$$

$$= \frac{a}{n-1}\int_0^\infty dx\,x^{n-1}\epsilon^{-ax}\sin rx - \frac{r}{n-1}\int_0^\infty dx\,x^{n-1}\epsilon^{-ax}\cos rx.$$

When $a=0$, this gives

$$\int_0^\infty dx\,x^{n-2}\sin rx = \frac{r}{1-n}\int_0^\infty dx\,x^{n-1}\cos rx = \frac{1}{r^{n-1}}\,\frac{\Gamma(n)}{1-n}\cos n\frac{\pi}{2}.$$

$$(p)\quad \text{If } n=\tfrac{1}{2},\quad \int_0^\infty \frac{dx}{x^{\frac{3}{2}}}\sin rx = (2r\pi)^{\frac{1}{2}}.$$

If in formulæ (l) and (m) we assume $x = z^{\frac{1}{n}}$, they become

$$\int_0^\infty dz\cos\left(rz^{\frac{1}{n}}\right) = \frac{n\Gamma(n)}{r^n}\cos\left(n\frac{\pi}{2}\right),$$

$$\int_0^\infty dz\sin\left(rz^{\frac{1}{n}}\right) = \frac{n\Gamma(n)}{r^n}\sin\left(n\frac{\pi}{2}\right).$$

Hence if $n=\frac{1}{2}$,

(q) $$\int_0^\infty dz \cos rz^2 = \tfrac{1}{2}\left(\frac{\pi}{2r}\right)^{\frac{1}{2}} = \int_0^\infty dz \sin rz^2.$$

The formulæ in this article are due principally to Euler, *Calc. Integ.* Vol. IV. p. 337. See also Mascheroni, *Adnotationes*, p. 53. Laplace, *Jour. de l'Ecole Polyt.* Cah. XV. p. 248, and Plana, *Mémoires de Bruxelles*, Vol. X.

(9) To find the value of $u=\int_0^\infty dx\, \epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)}$.

Differentiating with regard to a we find

$$\frac{du}{da} = -2a\int_0^\infty \frac{dx}{x^2}\,\epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)}.$$

Put $\dfrac{a}{x}=z$, the corresponding limits being

$$x=0,\ z=\infty;\ x=\infty,\ z=0.$$

Hence $$\frac{du}{da} = -2u;$$

this is a linear equation, the integral of which is

$$u = C\epsilon^{-2a}.$$

To determine the arbitrary constant, make $a=0$, when

$$C=\int_0^\infty dx\,\epsilon^{-x^2}=\frac{\pi^{\frac{1}{2}}}{2}.$$

(a) Hence $$\int_0^\infty dx\,\epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)} = \frac{\pi^{\frac{1}{2}}}{2}\epsilon^{-2a}.$$

This integral was first given by Laplace, *Mémoires de l'Institut*, 1810.

From this may be deduced the following integrals:

(b) $$\int_0^\infty dx\cos\left(x^2+\frac{a^2}{x^2}\right) = \frac{\pi^{\frac{1}{2}}}{2}\cos\left(\frac{\pi}{4}+2a\right)$$

$$= \tfrac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}(\cos 2a - \sin 2a).$$

(c) $$\int_0^\infty dx \sin\left(x^2 + \frac{a^2}{x^2}\right) = \frac{\pi^{\frac{1}{2}}}{2} \sin\left(\frac{\pi}{4} + 2a\right)$$

$$= \tfrac{1}{2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}} (\cos 2a + \sin 2a).$$

(d) $$\int_0^\infty dx\, \epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta} \cos\left\{\left(x^2 + \frac{a^2}{x^2}\right)\sin\theta\right\}$$

$$= \frac{\pi^{\frac{1}{2}}}{2} \epsilon^{-2a\cos\theta} \cos\left(2a \sin\theta + \frac{\theta}{2}\right).$$

(e) $$\int_0^\infty dx\, \epsilon^{-\left(x^2+\frac{a^2}{x^2}\right)\cos\theta} \sin\left\{\left(x^2 + \frac{a^2}{x^2}\right)\sin\theta\right\}$$

$$= \frac{\pi^{\frac{1}{2}}}{2} \epsilon^{-2a\cos\theta} \sin\left(2a \sin\theta + \frac{\theta}{2}\right).$$

Cauchy, *Mémoires des Savans Etrangers*, Vol. I. p. 638.

(10) Find the value of $\int_0^\infty dx\, \epsilon^{-a^2x^2} \cos 2rx$.

Calling the definite integral u, and differentiating with respect to r, we have

$$\frac{du}{dr} = -2\int_0^\infty dx\, x\, \epsilon^{-a^2x^2} \sin 2rx.$$

On integrating the second side with respect to x by parts, the equation becomes

$$\frac{du}{dr} = -\frac{2r}{a}u,$$

since the integrated part vanishes at both limits, and the unintegrated part when taken between the limits is equal to u. This equation on integration gives

$$u = C\epsilon^{-\frac{r^2}{a^2}}.$$

To determine the arbitrary constant, put $r = 0$,

then $$C = \int_0^\infty dx\, \epsilon^{-a^2x^2} = \frac{\pi^{\frac{1}{2}}}{2a}; \text{ so that}$$

(a) $$u = \int_0^\infty dx\, \epsilon^{-a^2x^2} \cos 2rx = \frac{\pi^{\frac{1}{2}}}{2a} \epsilon^{-\frac{r^2}{a^2}}.$$

Laplace, *Mémoires de l'Institut*, 1810, p. 290.

From this may be deduced the following integrals.

(*b*) $\int_0^\infty dx \cos a^2x^2 \cos 2rx = \frac{\pi^{\frac{1}{2}}}{2a} \cos\left(\frac{\pi}{4} - \frac{r^2}{a^2}\right).$

(*c*) $\int_0^\infty dx \sin a^2x^2 \cos 2rx = \frac{\pi^{\frac{1}{2}}}{2a} \sin\left(\frac{\pi}{4} - \frac{r^2}{a^2}\right).$

Fourier, *Traité de la Chaleur*, p. 533.

(11) To find the value of $u = \int_0^\infty dx \frac{\cos ax}{1 + x^2}$.

Differentiating twice with respect to a, we have

$$\frac{d^2u}{da^2} = -\int_0^\infty dx \frac{x^2 \cos ax}{1 + x^2} = -\int_0^\infty dx \cos ax + u.$$

By formula (*g*) of Ex. (8) $\int_0^\infty dx \cos ax = 0$; therefore

$$\frac{d^2u}{da^2} - u = 0.$$

The integral of this is

$$u = C\epsilon^a + C_1\epsilon^{-a}.$$

To determine the constants, we observe that u cannot increase continually with a, and therefore the term involving ϵ^a must vanish, or $C = 0$. This being the case we have, when $a = 0$,

$$u = C_1 = \int_0^\infty \frac{dx}{1 + x^2} = \frac{\pi}{2}.$$

Therefore (*a*) $\int_0^\infty \frac{dx \cos ax}{1 + x^2} = \frac{\pi}{2} \epsilon^{-a}.$

On differentiating this with respect to a, there results

(*b*) $\int_0^\infty dx \frac{x \sin ax}{1 + x^2} = \frac{\pi}{2} \epsilon^{-a}.$

Integrating with respect to a and determining the constant so as to make the integral vanish with a, we find

(*c*) $\int_0^\infty \frac{dx}{x} \frac{\sin ax}{1 + x^2} = \frac{\pi}{2} (1 - \epsilon^{-a}).$

Laplace, *Mémoires de l'Académie*, 1782.

It is to be observed that the formula (c) is discontinuous, as the integral is equal to $\frac{1}{2}\pi\epsilon^{-a}$ when a is positive, and to $\frac{1}{2}\pi\epsilon^{a}$ when a is negative. Libri* has accordingly expressed the value of the integral in the following manner:

$$\int_0^\infty dx \frac{\cos ax}{1+x^2} = \frac{\pi}{2}\left(\frac{\epsilon^a}{1+0^{-a}} + \frac{\epsilon^{-a}}{1+0^a}\right).$$

By a similar method we find

$$(d) \qquad \int_0^\infty dx \frac{\cos ax}{1+x^4} = \frac{\pi}{2^{\frac{3}{2}}}\epsilon^{-\frac{a}{2^{\frac{1}{2}}}}\left(\cos\frac{a}{2^{\frac{1}{2}}} + \sin\frac{a}{2^{\frac{1}{2}}}\right).$$

Laplace, *Mémoires de l'Institut*, 1810, p. 265.

For the value of $\int_0^\infty dx \frac{\cos ax}{1+x^{2n}}$ see *Jour. de l'Ecole Polyt.* Cah. XVI. p. 225 (Poisson), and for that of $\int_0^\infty dx \frac{\cos ax}{(1+x^2)^{2n}}$ see *Jour. de Mathématiques*, Vol. v. p. 110 (Catalan).

(12) To find the value of $\int_0^\pi \frac{dx \cos rx}{1-2a\cos x+a^2}$, $a<1$.

If we expand the denominator we have the series

$$\frac{1}{1-a^2}(1 + 2a\cos x + 2a^2\cos 2x + \&c. + 2a^r\cos rx + \&c.)$$

Multiply by $dx \cos rx$ and integrate: every term vanishes at both limits except

$2a^r\int_0^\pi dx(\cos rx)^2 = a^r\int_0^\pi dx(1+\cos 2rx) = \pi a^r$; therefore

$$(a) \qquad \int_0^\pi \frac{dx\cos rx}{1-2a\cos x+a^2} = \frac{\pi a^r}{1-a^2}.$$

Euler, *Calc. Integ.* Vol. IV. Sup. 5.

For the general expressions for

$\int_0^\pi \frac{dx\cos rx}{(1-2a\cos x+a^2)^n}$ and $\int_0^\pi dx\cos rx\,(1-2a\cos x+a^2)^n$,

the reader may consult Legendre, *Exercices*, Vol. I. p. 373.

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[*] *Crelle's Journal*, Vol. X. p. 309.

By a similar expansion it is easily seen that

(*b*) $\int_0^\pi dx \log(1-2a\cos x+a^2)=0$, or $2\pi\log a$,

according as a is less or greater than 1.

Poisson, *Journal de l'Ecole Polytechnique*, Cah. XVII. p. 617.

Also in like manner we find

(*c*) $\int_0^\pi dx \cos rx \log(1-2a\cos x+a^2) = -\frac{\pi}{r}a^r$ or $-\frac{\pi}{r}a^{-r}$,

according as $a <$ or > 1.

Integrating (*b*) by parts, we find

(*d*) $\displaystyle\int_0^\pi \frac{dx\, x\sin x}{1-2a\cos x+a^2} = \frac{\pi}{a}\log(1+a)$ or $= \displaystyle\frac{\pi}{a}\log\left(1+\frac{1}{a}\right)$,

according as $a <$ or > 1.

Integrating (*c*) by parts, we find

(*e*) $\displaystyle\int_0^\pi \frac{dx \sin x \sin rx}{1-2a\cos x+a^2} = \frac{\pi}{2}a^{r-1}$ or $= \displaystyle\frac{\pi}{2}a^{-(r+1)}$,

according as $a <$ or > 1.

(13) To find the value of $\displaystyle\int_0^\infty \frac{dx}{1+x^2}\cdot\frac{1}{1-2a\cos rx+a^2}$, $a<1$.

Expand the second factor as before, and integrate each term separately by (11, *a*); then on summing the result, we have

(*a*) $\displaystyle\int_0^\infty \frac{dx}{1+x^2}\,\frac{1}{1-2a\cos rx+a^2} = \frac{\pi}{2}\,\frac{1}{1-a^2}\,\frac{1+a\epsilon^{-r}}{1-a\epsilon^{-r}}$.

In like manner we find

(*b*) $\displaystyle\int_0^\infty \frac{dx}{1+x^2}\log(1-2a\cos rx+a^2) = \pi\log(1-a\epsilon^{-r})$.

Also it is easily seen that

$$\frac{\sin rx}{1-2a\cos rx+a^2} = \sin rx + a\sin 2rx + a^2\sin 3rx + \&c.$$

Hence multiplying by $\frac{x\,dx}{1+x^2}$ and integrating from 0 to ∞ by (11, *b*), we find

$$(c)\quad \int_0^\infty \frac{dx\,x}{1+x^2}\,\frac{\sin rx}{1-2a\cos rx+a^2}=\frac{1}{2}\,\frac{\pi}{\epsilon^r-a}.$$

In (*b*) make $a=1$, then

$$(d)\quad \int_0^\infty \frac{dx}{1+x^2}\log\left(\sin\frac{rx}{2}\right)=\frac{\pi}{2}\log\left(\frac{1-\epsilon^{-r}}{2}\right).$$

Changing the sign of a and then making $a=1$, we have

$$(e)\quad \int_0^\infty \frac{dx}{1+x^2}\log\left(\cos\frac{rx}{2}\right)=\frac{\pi}{2}\log\left(\frac{1+\epsilon^{-r}}{2}\right).$$

Subtracting the second of these from the first,

$$(f)\quad \int_0^\infty \frac{dx}{1+x^2}\log\left(\tan\frac{rx}{2}\right)=\frac{\pi}{2}\log\left(\frac{\epsilon^r-1}{\epsilon^r+1}\right).$$

In (*c*) make $a=1$; then

$$(g)\quad \int_0^\infty \frac{dx\,x}{1+x^2}\cot\frac{rx}{2}=\frac{\pi}{\epsilon^r-1}.$$

Changing the sign of a and then making $a=1$, we have

$$(h)\quad \int_0^\infty \frac{dx\,x}{1+x^2}\tan\frac{rx}{2}=\frac{\pi}{\epsilon^r+1}.$$

The formulæ (*a*), (*b*), (*c*) are due to Legendre: see his *Exercices*, Vol. II. p. 123.

The formulæ (*d*), (*e*), (*f*), (*g*), (*h*) were first given by Georges Bidone, in the *Mémoires de Turin*, Vol. XX.

(14) To find the value of

$$\int_0^{\frac{1}{2}\pi} dx\log(\sin x)=\int_0^{\frac{1}{2}\pi} dx\log(\cos x).$$

By Cotes's Theorem we have

$$\frac{x^{2n}-1}{x^2-1}=\left(x^2-2x\cos\frac{1}{n}\pi+1\right)\ldots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right).$$

Let $x = 1$; then as $\dfrac{x^{2n}-1}{x^2-1} = n$ when $x = 1$, we have

$$n = 2^{2(n-1)} \sin^2 \frac{1}{n}\frac{\pi}{2} \dots \sin^2 \frac{n-1}{n}\frac{\pi}{2}.$$

Take the logarithms on both sides, and divide by n; then

$$\frac{\log n - 2(n-1)\log 2}{2n} = \left(\log \sin \frac{1}{n}\frac{\pi}{2} + \dots + \log \sin \frac{n-1}{n}\frac{\pi}{2}\right)\frac{1}{n}.$$

Let n become infinite and equal to $\dfrac{1}{d\theta}$. The first side becomes $-\log 2 = \log \frac{1}{2}$, as $\dfrac{\log n}{n} = 0$ when $n = \dfrac{1}{0}$; and the second side is transformed into the definite integral

$$\int_0^1 \log(\sin \tfrac{1}{2}\theta\pi)\, d\theta; \text{ therefore}$$

$$\int_0^1 \log(\sin \tfrac{1}{2}\theta\pi)\, d\theta = \log \tfrac{1}{2}.$$

Putting $x = \frac{1}{2}\theta\pi$, this is equivalent to

$$(a) \qquad \int_0^{\frac{1}{2}\pi} dx \log(\sin x) = \tfrac{1}{2}\pi \log(\tfrac{1}{2}).$$

This demonstration of a theorem due to Euler* is given by Mr Leslie Ellis in the *Cam. Math. Journal*, Vol. II. p. 282.

If we put $\sin x = y$ in (*a*), it becomes

$$(b) \qquad \int^1 \frac{dy \log y}{(1-y^2)^{\frac{1}{2}}} = \tfrac{1}{2}\pi \log(\tfrac{1}{2}).$$

On integrating (*a*) by parts, we find

$$(c) \qquad \int_0^{\frac{1}{2}\pi} dx\, x \cot x = \tfrac{1}{2}\pi \log 2.$$

On integrating (*c*) by parts, we find

$$(d) \quad \int_0^{\frac{1}{2}\pi} \frac{dx\, x^2}{(\sin x)^2} = \pi \log 2 = \int_0^\infty dy\, (\cot^{-1} y)^2,$$

if we put $x = \cot^{-1} y$.

(15) Integrate $\displaystyle\int \frac{dx\, x \log x}{(1-x^2)^{\frac{1}{2}}}$ by parts; then

$$\int \frac{dx\, x \log x}{(1-x^2)^{\frac{1}{2}}} = (1-x^2)^{\frac{1}{2}} - \log\{1 + (1-x^2)^{\frac{1}{2}}\} + \{1 - (1-x^2)^{\frac{1}{2}}\} \log x.$$

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* *Acta Petrop.* Vol. I. p. 2.

On taking this between the limits 0 and 1, we find since $\{1-(1-x^2)^{\frac{1}{2}}\}\log x$ vanishes at both limits,

$$(a) \qquad \int_0^1 \frac{dx\, x \log x}{(1-x^2)^{\frac{1}{2}}} = \log 2 - 1.$$

Also on integrating by parts, we have

$$\int \frac{dx\, x^2 \log x}{(1-x^2)^{\frac{1}{2}}} = -\frac{x(1-x^2)^{\frac{1}{2}}}{2} \log x + \tfrac{1}{2}\int dx\,(1-x^2)^{\frac{1}{2}} + \tfrac{1}{2}\int \frac{dx \log x}{(1-x^2)^{\frac{1}{2}}};$$

and therefore taking it between the limits 0 and 1,

$$(b) \qquad \int_0^1 \frac{dx\, x^2 \log x}{(1-x^2)^{\frac{1}{2}}} = \frac{\pi}{4}\left\{\log\left(\tfrac{1}{2}\right) + 1\right\}.$$

Euler, *Nov. Com. Petrop.* Vol. XIX. p. 30.

(16) To find the value of $\int_0^1 \frac{dx \log x}{1+x}$.

Since $$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \&c.,$$

and $$\int dx\, x^n \log x = \frac{x^{n+1} \log x}{n-1} - \frac{x^{n+1}}{(n+1)^2};$$

which, when taken between the limits 0 and 1, is reduced to

$$\int_0^1 dx\, x^n \log x = -\frac{1}{(n+1)^2},$$

it follows that

$$(a) \qquad \int_0^1 \frac{dx \log x}{1+x} = -\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \&c.\right) = -\frac{\pi^2}{12}.$$

In a similar way we may find

$$(b) \qquad \int_0^1 \frac{dx \log x}{1-x} = -\frac{\pi^2}{6},$$

$$(c) \quad \int_0^1 \frac{dx \log x}{1-x^2} = -\frac{\pi^2}{8}, \qquad (d) \quad \int_0^1 \frac{dx\, x \log x}{1-x^2} = -\frac{\pi^2}{24}.$$

Euler, *Ib.*

(17) The integral $\int_0^\pi \frac{dx\,(\sin x)^{2n}}{(1-2a\cos x + a^2)^n}$ by differentiation with respect to a leads to the equation

$$\frac{d^2y}{da^2} + \frac{2n+1}{a}\frac{dy}{da} = 0.$$

From which we find

$$\int_0^\pi \frac{dx\,(\sin x)^{2n}}{(1-2a\cos x+a^2)^n} = \frac{(2n-1)(2n-3)\dots 3\,.\,1}{2n(2n-2)(2n-4)\dots 4\,.\,2}\,\frac{\pi}{2},$$

when $a < 1$. If $a > 1$, the only difference is that the preceding result is to be multiplied by a^{-2n}.

Poisson, *Journal de l'Ecole Polytechnique*, Cah. XVII. p. 614.

(18) To find the value of $\int_0^\pi \frac{dx\,x\sin x}{1+(\cos x)^2}$.

On expanding the denominator, we have a series consisting of the even powers of $\cos x$. Take one of these as $(\cos x)^{2r}$: then

$$\int dx\,x\sin x\,(\cos x)^{2r} = -\frac{x(\cos x)^{2r+1}}{2r+1} + \frac{1}{2r+1}\int(\cos x)^{2r+1}\,dx,$$

by integration by parts. In taking the limits between 0 and π $\int_0^\pi dx\,(\cos x)^{2r+1}$ vanishes as $2r+1$ is odd; therefore

$$\int_0^\pi dx\,x\sin x\,(\cos x)^{2r} = \frac{\pi}{2r+1},$$

and $\int_0^\pi dx\,\frac{x\sin x}{1+(\cos x)^2} = \pi\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\&\text{c.}\right) = \frac{\pi^2}{4}$.

Poisson, *Ib.* p. 623.

(19) To find the value of $\int_0^\infty \frac{dx}{1+x^2}\,\frac{\sin ax}{\sin bx}$.

There are here three cases to be considered—according as a is less than b, equal to b or some multiple of b, and greater than b, not being a multiple of it.

Let $a < b$, and let $\frac{a}{b}\pi = \theta$; then if $\frac{\sin ax}{\sin bx}$ be decomposed into quadratic factors, we find

$$\frac{\sin ax}{\sin bx} = 2\pi\left(\frac{\sin\theta}{\pi^2-b^2x^2} - \frac{2\sin 2\theta}{4\pi^2-b^2x^2} + \frac{3\sin 3\theta}{9\pi^2-b^2x^2} - \&\text{c.}\right).$$

We have therefore to integrate a series of functions of the form

$$A\int_0^\infty \frac{dx}{1+x^2}\,\frac{1}{n^2\pi^2-b^2x^2}.$$

Now if we decompose $\frac{1}{(1+x^2)(m^2-x^2)}$ into quadratic factors and integrate from 0 to ∞, observing that we have $\int_0^\infty \frac{dx}{m^2-x^2} = 0$, there results

$$\int_0^\infty \frac{dx}{(1+x^2)(m^2-x^2)} = \frac{\pi}{2}\,\frac{1}{m^2+1}.$$

Hence if we have a function of x which can be decomposed into a sum of partial fractions of the form $\frac{1}{m^2-x^2}$, so that

$$f(x) = \Sigma\,\frac{A_n}{m_n^2-x^2},$$

it appears that

$$\int_0^\infty \frac{dx}{1+x^2}\,f(x) = \frac{\pi}{2}\,\Sigma\,\frac{A_n}{m_n^2+1} = \frac{\pi}{2}\,f\{(-1)^{\frac{1}{2}}\},$$

-1 being substituted for x^2.

In the case under consideration, therefore, we have

$$(a)\qquad \int_0^\infty \frac{dx}{1+x^2}\,\frac{\sin ax}{\sin bx} = \frac{\pi}{2}\,\frac{\epsilon^a-\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}}.$$

If a be greater than b and not a multiple of it, let $a = 2rb + c$ where r is an integer, and $c < a$. Then as

$$\sin(2rb+c)x - \sin\{2(r-1)b+c\}x = 2\sin bx\cos\{(2r-1)b+c\}x,$$

we have

$$\frac{\sin ax}{\sin bx} = \frac{\sin\{2(r-1)b+c\}x}{\sin bx} + 2\cos(a-b)x.$$

Similarly,

$$\frac{\sin\{2(r-1)b+c\}x}{\sin bx} = \frac{\sin\{2(r-2)b+c\}x}{\sin bx} + 2\cos(a-3b),$$

....

$$\frac{\sin(2b+c)x}{\sin bx} = \frac{\sin cx}{\sin bx} + 2\cos(b+c)x:$$

so that

$$\frac{\sin ax}{\sin bx} = \frac{\sin cx}{\sin bx} + 2\{\cos(a-b)x + \cos(a-3b)x + \ldots + \cos(b+c)x\}.$$

Now by what has preceded, since $c<b$,

$$\int_0^\infty \frac{dx}{1+x^2}\,\frac{\sin cx}{\sin bx}=\frac{\pi}{2}\,\frac{\epsilon^c-\epsilon^{-c}}{\epsilon^b-\epsilon^{-b}};$$

also, by (11, a), we have

$$\int_0^\infty \frac{dx}{1+x^2}\cos mx=\frac{\pi}{2}\,\epsilon^{-m}.$$

Applying this to each term within the brackets and summing the series, we find

$$(b)\quad \int_0^\infty \frac{dx}{1+x^2}\,\frac{\sin ax}{\sin bx}=\frac{\pi}{2}\,\frac{\epsilon^c-\epsilon^{-c}}{\epsilon^b-\epsilon^{-b}}+\frac{\pi(\epsilon^{-c}-\epsilon^{-a})}{\epsilon^b-\epsilon^{-b}}=\frac{\pi}{2}\,\frac{\epsilon^c+\epsilon^{-c}-2\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}},$$

where $a=2rb+c$.

If a be a multiple of b, $c=0$ and $\dfrac{\sin ax}{\sin bx}$ is reduced to the finite series within the brackets, so that

$$(c)\quad \int_0^\infty \frac{dx}{1+x^2}\,\frac{\sin ax}{\sin bx}=\pi\,\frac{1-\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}}.$$

In the same way it will be found that

$$(d)\quad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\cos ax}{\sin bx}=\frac{\pi}{2}\,\frac{\epsilon^a+\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}},\quad a<b,$$

$$(e)\quad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\cos ax}{\sin bx}=\frac{\pi}{2}\,\frac{\epsilon^c-\epsilon^{-c}+2\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}},\quad a=2rb+c,$$

$$(f)\quad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\cos ax}{\sin bx}=\frac{\pi\epsilon^{-a}}{\epsilon^b-\epsilon^{-b}},\quad a=(2r+1)\,b.$$

In a similar manner also are found the following integrals:

$$(g)\quad \int_0^\infty \frac{dx}{x}\,\frac{1}{1+x^2}\,\frac{\sin ax}{\cos bx}=\frac{\pi}{2}\,\frac{\epsilon^a-\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}},\quad a<b,$$

$$(h)\quad \int_0^\infty \frac{dx}{x}\,\frac{1}{1+x^2}\,\frac{\sin ax}{\cos bx}=$$

$$\frac{\pi}{2}\left\{1-(-)^r\right\}+(-)^r\,\frac{\pi}{2}\,\frac{\epsilon^c+\epsilon^{-c}}{\epsilon^b+\epsilon^{-b}}-\frac{\pi\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}},\quad a=2rb+c.$$

Also,

$$(k)\quad \int_0^\infty \frac{dx}{1+x^2}\,\frac{\cos ax}{\cos bx} = \frac{\pi}{2}\,\frac{\epsilon^a+\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}},\ a<b,$$

$$(l)\quad \int_0^\infty \frac{dx}{1+x^2}\,\frac{\cos ax}{\cos bx} = (-)^r\frac{\pi}{2}\,\frac{\epsilon^c-\epsilon^{-c}}{\epsilon^b+\epsilon^{-b}} + \frac{\pi\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}},\ a=2rb+c.$$

Differentiating the last of these equations with respect to a and observing that $dc = da$, we have

$$(m)\quad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\sin ax}{\cos bx} = -(-)^r\frac{\pi}{2}\,\frac{\epsilon^c+\epsilon^{-c}}{\epsilon^b+\epsilon^{-b}} + \frac{\pi\epsilon^{-a}}{\epsilon^b+\epsilon^{-b}}.$$

Adding this to (h), we find

$$(n)\quad \int_0^\infty \frac{dx}{x}\,\frac{\sin ax}{\cos bx} = \frac{\pi}{2}\{1-(-)^r\}.$$

In this equation there is implied the condition that a shall not be an odd multiple of b. If $a = (2r+1)\,b$,

$$(p)\quad \int_0^\infty \frac{dx}{x}\,\frac{\sin ax}{\cos bx} = \frac{\pi}{2};$$

when $a = b$, we have

$$(q)\quad \int_0^\infty \frac{dx}{x}\tan ax = \frac{\pi}{2}.$$

The preceding remarkable integrals were first given by Cauchy (*Mémoires des Savans Etrangers*, Vol. I.): the demonstrations are taken from Legendre, *Exercices*, Vol. II. p. 174.

(20) To find the value of $\int_0^\infty \frac{dx\,(\epsilon^{ax}+\epsilon^{-ax})}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\sin rx$; when $a<\pi$.

By expanding the denominator, we have

$$\frac{1}{\epsilon^{\pi x}-\epsilon^{-\pi x}} = \epsilon^{-\pi x} + \epsilon^{-3\pi x} + \epsilon^{-5\pi x} + \&c.$$

Hence on multiplying by $(\epsilon^{ax}+\epsilon^{-ax})\sin rx$, we have to integrate two series of terms of the forms

$$\epsilon^{-\{(2n+1)\pi-a\}x}\sin rx,\ \text{and}\ \epsilon^{-\{(2n+1)\pi+a\}x}\sin rx;$$

the values of which are by (8, a and b),

$$\frac{r}{r^2+\{(2n+1)\pi-a\}^2}, \text{ and } \frac{r}{r^2+\{(2n+1)\pi+a\}^2}.$$

Hence we have

$$\int_0^\infty \frac{dx\,(\epsilon^{ax}+\epsilon^{-ax})}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\sin rx = r\left\{\frac{1}{r^2+(\pi-a)^2}+\frac{1}{r^2+(3\pi-a)^2}+\&\text{c.}\right\}$$

$$+\,r\left\{\frac{1}{r^2+(\pi+a)^2}+\frac{1}{r^2+(3\pi+a)^2}+\&\text{c.}\right\}.$$

Now if we decompose $\epsilon^r+2\cos a+\epsilon^{-r}$ into its quadratic factors, we have

$$\epsilon^r+2\cos a+\epsilon^{-r}=4\left(\sin\frac{a}{2}\right)^2\left\{1+\left(\frac{r}{\pi-a}\right)^2\right\}\times$$

$$\left\{1+\left(\frac{r}{\pi+a}\right)^2\right\}\left\{1+\left(\frac{r}{3\pi-a}\right)^2\right\}\ \&\text{c.}$$

Taking the logarithmic differential of this with respect to r we have

$$\frac{\epsilon^r-\epsilon^{-r}}{\epsilon^r+2\cos a+\epsilon^{-r}}=2r\left\{\frac{1}{r^2+(\pi-a)^2}+\frac{1}{r^2+(3\pi-a)^2}+\&\text{c.}\right\}$$

$$+\,2r\left\{\frac{1}{r^2+(\pi+a)^2}+\frac{1}{r^2+(3\pi+a)^2}+\&\text{c.}\right\};$$

therefore

$$(a)\quad \int_0^\infty \frac{dx\,(\epsilon^{ax}+\epsilon^{-ax})}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\sin rx=\frac{1}{2}\,\frac{\epsilon^r-\epsilon^{-r}}{\epsilon^r+2\cos a+\epsilon^{-r}}.$$

In like manner we find

$$(b)\quad \int_0^\infty \frac{dx\,(\epsilon^{ax}-\epsilon^{-ax})}{\epsilon^{\pi x}-\epsilon^{-\pi x}}\cos rx=\frac{\sin a}{\epsilon^r+2\cos a+\epsilon^{-r}}.$$

In (a) make $a=0$, then

$$(c)\quad \int_0^\infty \frac{dx\sin rx}{\epsilon^{\pi x}-\epsilon^{-\pi x}}=\frac{1}{4}\,\frac{\epsilon^r-1}{\epsilon^r+1}.$$

In (b) make $r=0$, then

$$(d)\quad \int_0^\infty \frac{dx\,(\epsilon^{ax}-\epsilon^{-ax})}{\epsilon^{\pi x}-\epsilon^{-\pi x}}=\frac{1}{2}\tan\frac{a}{2}.$$

Differentiating (c) with respect to r,

$$(e)\qquad \int_0^\infty \frac{dx\, x\cos rx}{\epsilon^{\pi x}-\epsilon^{-\pi x}}=\frac{1}{2}\frac{\epsilon^r}{(\epsilon^r+1)^2}.$$

(21) By a process similar to that in (20) we find

$$(a)\qquad \int_0^\infty \frac{dx\,(\epsilon^{ax}+\epsilon^{-ax})}{\epsilon^{\pi x}+\epsilon^{-\pi x}}\cos rx=\frac{(\epsilon^{\frac{1}{2}r}+\epsilon^{-\frac{1}{2}r})\cos\frac{1}{2}a}{\epsilon^r+\epsilon^{-r}+2\cos a}.$$

In all these integrals a is less than π.

If $a=0$,

$$(b)\qquad \int_0^\infty \frac{dx\cos rx}{\epsilon^{\pi x}+\epsilon^{-\pi x}}=\frac{1}{2}\frac{\epsilon^{\frac{1}{2}r}}{\epsilon^r+1}=\frac{1}{2}\frac{1}{\epsilon^{\frac{1}{2}r}+\epsilon^{-\frac{1}{2}r}}.$$

Multiply both sides by $\epsilon^{-mr}dr$, and integrate from $r=0$ to $r=\infty$: then as

$$\int_0^\infty dr\,\epsilon^{-mr}\cos rx=\frac{m}{m^2+x^2},$$

$$(c)\qquad \int_0^\infty \frac{dx}{(m^2+x^2)(\epsilon^{\pi x}+\epsilon^{-\pi x})}=\frac{1}{2m}\int\frac{dr\,\epsilon^{-mr}}{\epsilon^{\frac{1}{2}r}+\epsilon^{-\frac{1}{2}r}}.$$

Putting $\epsilon^{-r}=z$, the second side becomes

$$\frac{1}{2m}\int_0^1\frac{dz\,z^{m-\frac{1}{2}}}{1+z}.$$

Hence when $m=\frac{1}{2}$,

$$(d)\qquad \int_0^\infty \frac{dx}{(\frac{1}{4}+x^2)(\epsilon^{\pi x}+\epsilon^{-\pi x})}=\log 2;$$

when $m=1$,

$$(e)\qquad \int_0^\infty \frac{dx}{(1+x^2)(\epsilon^{\pi x}+\epsilon^{-\pi x})}=1-\frac{1}{4}\pi.$$

(22) Poisson* has demonstrated the following formulæ:

If $u=\cos x+(-)^{\frac{1}{2}}\sin x$, $v=\cos x-(-)^{\frac{1}{2}}\sin x$,

so that $u^n+v^n=2\cos nx$, $u^n-v^n=2(-)^{\frac{1}{2}}\sin nx$;

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* *Journal de l'Ecole Polytechnique*, Cah. XIX. p. 482.

then $$\int_0^\pi \frac{dx\,(1-p\cos x)}{1-2p\cos x+p^2}\{F(a+v)+F(a+u)\}$$
$$=\pi\{F(a+p)+F(a)\}\ldots\ldots\ldots\ldots(1),$$

$$\int_0^\pi \frac{dx\sin x}{1-2p\cos x+p^2}\{F(a+v)-F(a+u)\}$$
$$=\frac{\pi}{p\,(-)^{\frac{1}{2}}}\{F(a+p)-F(a)\}\ldots\ldots\ldots(2).$$

These expressions may be easily proved by developing the functions and the denominator in terms of cosines and of sines of multiples of x, integrating each term separately, and observing that

$$\int_0^\pi dx\cos mx\cos nx=0,\quad \int_0^\pi dx\sin mx\sin nx=0,$$

$$\int_0^\pi dx\,(\cos nx)^2=\frac{\pi}{2},\quad \int_0^\pi dx\,(\sin nx)^2=\frac{\pi}{2}.$$

In applying these formulæ it is to be observed, (1) that p must be less than 1; (2) that for the particular value assigned to a none of the differential coefficients of $F(a)$ become infinite; (3) that the sum and difference of $F(a+v)$ and $F(a+u)$ be expanded in converging series; (4) that the function under the sign of integration should not for any value of x between 0 and π become infinite, while the corresponding series remains finite, or *vice versa*.

From the equation (1) may be readily derived the following:

$$\int_0^\pi \frac{dx\,F(a+v)+F(a+u)}{1-2p\cos x+p^2}=\frac{2\pi}{1-p^2}F(a+p)\ldots\ldots(3).$$

In equation (3) put $F(a)=\epsilon^{ca}$, c being a constant. Then

$$F(a+v)=\epsilon^{ca}\,\epsilon^{c\cos x}\{\cos(c\sin x)-(-)^{\frac{1}{2}}\sin(c\sin x)\},$$
$$F(a+u)=\epsilon^{ca}\,\epsilon^{c\cos x}\{\cos(c\sin x)+(-)^{\frac{1}{2}}\sin(c\sin x)\}.$$

Therefore

(*a*) $$\int_0^\pi \frac{dx\,\epsilon^{c\cos x}\cos(c\sin x)}{1-2p\cos x+p^2}=\frac{\pi}{1-p^2}\epsilon^{cp}.$$

In the same way from (2) we have

$$(b)\quad \int_0^{\pi} \frac{dx\, e^{c\cos x} \sin(c\sin x)\sin x}{1-2p\cos x+p^2} = \frac{\pi}{2p}(e^{cp}-1).$$

If we expand both sides of these equations, and equate the coefficients of like powers of p, we have

$$(c)\quad \textstyle\int_0^{\pi} dx\, e^{c\cos x} \cos(c\sin x)\cos nx = \frac{\pi}{2}\,\frac{c^n}{1.2\ldots n}.$$

$$(d)\quad \textstyle\int_0^{\pi} dx\, e^{c\cos x} \sin(c\sin x)\sin nx = \frac{\pi}{2}\,\frac{c^n}{1.2\ldots n}.$$

Differentiating (a) and (b) r times with respect to c,

$$(e)\quad \int_0^{\pi} \frac{dx\, e^{c\cos x}\cos(c\sin x+rx)}{1-2p\cos x+p^2} = \frac{\pi p^r}{1-p^2}\,e^{cp}.$$

$$(f)\quad \int_0^{\pi} \frac{dx\, e^{c\cos x}\sin(c\sin x+rx)\sin x}{1-2p\cos x+p^2} = \frac{\pi}{2}\,p^{r-1}e^{cp}.$$

In equation (8) make $F(\alpha)=\alpha^a$, a being positive, and then make $\alpha=1$; this gives

$$F\{1+\cos x+(-)^{\frac{1}{2}}\sin x\} = \{1+\cos x+(-)^{\frac{1}{2}}\sin x\}^a$$

$$= 2^a\left(\cos\frac{x}{2}\right)^a\left\{\cos a\,\frac{x}{2}+(-)^{\frac{1}{2}}\sin\frac{ax}{2}\right\},$$

since $1+\cos x=2\cos^2\dfrac{x}{2}$, and $\sin x=2\sin\dfrac{x}{2}\cos\dfrac{x}{2}$.

Hence, putting x for $\frac{1}{2}x$ and therefore $\frac{1}{2}\pi$ for π at the limit,

$$(g)\quad \int_0^{\frac{1}{2}\pi} \frac{dx\,(\cos x)^a\cos ax}{1-2p\cos 2x+p^2} = \frac{\pi}{2(1-p^2)}\left(\frac{1+p}{2}\right)^a.$$

In this make $p=0$; then

$$(h)\quad \textstyle\int_0^{\frac{1}{2}\pi} dx\,(\cos x)^a\cos ax = \frac{\pi}{2}\,\frac{1}{2^a}.$$

Developing both sides of (g) and equating the coefficients of like powers of p, we find

$$(k)\quad \textstyle\int_0^{\frac{1}{2}\pi} dx(\cos x)^a\cos ax\cos 2rx = \frac{\pi}{4}\,\frac{a(a-1)\ldots a-r+1}{1.2.3\ldots r}\,\frac{1}{2^a}.$$

Differentiating (g) with respect to a and then making $a = 0$, we have

$$(l)\quad \int_0^{\frac{1}{2}\pi} \frac{dx \log(\cos x)}{1 - 2p\cos 2x + p^2} = \frac{\pi}{2(1-p^2)} \log\left(\frac{1+p}{2}\right).$$

Expanding this last, and equating the coefficients of like powers of p,

$$(m)\quad \int_0^{\frac{1}{2}\pi} dx \log(\cos x) \cos 2rx = (-)^{r-1} \frac{\pi}{4} \frac{1}{r}.$$

(23) Swanberg[*] has proved the following theorems more general than those of Poisson. If

$$2M = f(a + \alpha u^\lambda,\ b + \beta u^\mu \ldots) + f(a + \alpha v^\lambda,\ b + \beta v^\mu \ldots),$$

$$2(-)^{\frac{1}{2}} N = f(a + \alpha u^\lambda,\ b + \beta u^\mu \ldots) - f(a + \alpha v^\lambda,\ b + \beta v^\mu \ldots),$$

u and v having the same signification as before; then

$$\int_0^\pi \frac{dx\, M}{1 - 2p\cos x + p^2} = \frac{\pi}{1 - p^2} f(a + \alpha p^\lambda,\ b + \beta p^\mu \ldots),$$

$$\int_0^\pi \frac{dx\, N \sin x}{1 - 2p\cos x + p^2} = \frac{\pi}{2r} f(a + \alpha p^\lambda,\ b + \beta p^\mu \ldots) - \frac{\pi}{2r} f(a, b \ldots).$$

Let $\qquad f(a, b \ldots) = a^l b^m \ldots,$

$$a = \alpha = b = \beta \ldots = 1.$$

Then, changing x into $2x$, and therefore the limit π into $\frac{1}{2}\pi$, we obtain from the first of these expressions

$$\int_0^{\frac{1}{2}\pi} \frac{dx (\cos \lambda x)^l (\cos \mu x)^m \ldots \cos(l\lambda + m\mu + \ldots) x}{1 - 2p\cos 2x + p^2}$$

$$= \frac{\pi}{2} \frac{1}{1 - p^2} \left(\frac{1 + p^\lambda}{2}\right)^l \left(\frac{1 + p^\mu}{2}\right)^m \ldots\ldots$$

(24) The same writer (p. 233) has proved the following theorems. If

$$P = f(a + \alpha u^\lambda,\ b + \beta u^\mu \ldots) + f(a + \alpha v^\lambda,\ b + \beta v^\mu \ldots),$$

$$(-)^{\frac{1}{2}} Q = f(a + \alpha u^\lambda,\ b + \beta u^\mu \ldots) - f(a + \alpha v^\lambda,\ b + \beta v^\mu \ldots),$$

where u and v have the same signification as before,

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[*] *Nova Acta Reg. Soc. Upsaliensis*, Vol. x. p. 271.

$$(1)\quad \int_0^\infty \frac{dx\, P}{h^2+x^2} = \frac{\pi}{h} f(a+\alpha\epsilon^{-\lambda h},\ b+\beta\epsilon^{-\mu h}\ldots),$$

$$(2)\quad \int_0^\infty \frac{dx\, x\, Q}{h^2+x^2} = \pi f(a+\alpha\epsilon^{-\lambda h},\ b+\beta\epsilon^{-\mu h}\ldots) - \pi f(a,b\ldots).$$

These expressions are easily proved by expansion, with the assistance of the formula in Ex. (11): they are evidently subject to the same cases of exception as the formulæ of Poisson.

Let $\qquad f(a,b\ldots) = a^l . b^m \ldots\ldots$

and $a=\alpha=b=\beta=1$; then changing λ into 2λ, μ into 2μ, &c., we have

$$(a)\quad \int_0^\infty \frac{dx(\cos\lambda x)^l(\cos\mu x)^m\ldots\cos(l\lambda+m\mu+\ldots)x}{h^2+x^2}$$

$$= \frac{\pi}{2h}\left(\frac{1+\epsilon^{-2\lambda h}}{2}\right)^l \left(\frac{1+\epsilon^{-2\mu h}}{2}\right)^m \ldots\ldots$$

$$(b)\quad \int_0^\infty \frac{dx\, x(\cos\lambda x)^l(\cos\mu x)^m\ldots\sin(l\lambda+m\mu+\ldots)x}{h^2+x^2}$$

$$= \frac{\pi}{2}\left(\frac{1+\epsilon^{-2\lambda h}}{2}\right)^l \left(\frac{1+\epsilon^{-2\mu h}}{2}\right)^m \ldots - \frac{\pi}{2}\,\frac{1}{\epsilon^{l+m+\ldots}}.$$

If in (a) m, &c. be made equal to zero, the expression becomes

$$(c)\quad \int_0^\infty \frac{dx(\cos\lambda x)^l \cos l\lambda x}{h^2+x^2} = \frac{\pi}{2h}\left(\frac{1+\epsilon^{-2\lambda h}}{2}\right)^l.$$

Let $f(a,b) = a^l . \epsilon^{mb}$, considering two terms only, and $a=\alpha=\beta=1$, $b=0$. Then as before changing λ into 2λ, we find by formula (1),

$$(d)\quad \int_0^\infty \frac{dx(\cos\lambda x)^l . \epsilon^{m\cos\mu x}\cos(l\lambda x + m\sin\mu x)}{h^2+x^2}$$

$$= \frac{\pi}{2h}\left(\frac{1+\epsilon^{-2\lambda h}}{2.}\right)^l \epsilon^{m\epsilon^{-\mu h}}.$$

In this expression put $\lambda = 0$, then

$$(e)\quad \int_0^\infty \frac{dx\,\epsilon^{m\cos\mu x}\cos(m\sin\mu x)}{h^2+x^2} = \frac{\pi}{2h}\epsilon^{m\epsilon^{-\mu h}}.$$

The student may exercise himself in deducing other integrals from the general formulæ by assuming other forms for the functions, and other values for the constants $a, b \ldots$ $\alpha, \beta \ldots$.

(25) Jacobi* has proved the following remarkable transformation of a definite integral:

$$\int_0^\pi dx\, f^{(r)}(\cos x)(\sin x)^{2r} = 1.3.5.7 \ldots (2r-1) \int_0^\pi dx\, f(\cos x) \cos rx;$$

where $f^{(r)}(z) = \left(\dfrac{d}{dz}\right)^r f(z)$, and all the differential coefficients up to the $(r-1)^{\text{th}}$ inclusive remain continuous from $z = 1$ to $z = -1$, or from $x = 0$ to $x = \pi$. To demonstrate this formula we must premise the following.

If $z = \cos x$,

$$\frac{d^{r-1}(1-z^2)^{\frac{2r-1}{2}}}{dz^{r-1}} = (-)^{r-1} 1.3.5.7 \ldots (2r-1) \frac{\sin rx}{r}.$$

This is easily proved by the formula (A) p. 16, for writing in it $r-1$ for r and $\dfrac{2r-1}{2}$ for n, and making $a = 1$, $b = 0$, $c = -1$, it becomes

$$\frac{d^{r-1}(1-z^2)^{\frac{2r-1}{2}}}{dz^{r-1}} = (-)^{r-1} 3.5 \ldots (2r-1) \Big\{ (1-z^2)^{\frac{1}{2}} z^{r-1}$$

$$- \frac{(r-1)(r-2)}{1.2.3}(1-z^2)^{\frac{3}{2}} z^{r-3} + \frac{(r-1)(r-2)(r-3)(r-4)}{1.2.3.4}(1-z^2)^{\frac{5}{2}} z^{r-5} - \&c. \Big\}.$$

Or, putting $\cos x$ for z and $\sin x$ for $(1-z^2)^{\frac{1}{2}}$,

$$\frac{d^{r-1}(1-z^2)^{\frac{2r-1}{2}}}{dz^{r-1}} = (-)^{r-1} 3.5 \ldots (2r-1) \Big\{ (\cos x)^{r-1} \sin x$$

$$- \frac{(r-1)(r-2)}{1.2.3}(\cos x)^{r-3}(\sin x)^3$$

$$+ \frac{(r-1)(r-2)(r-3)(r-4)}{1.2.3.4}(\cos x)^{r-5}(\sin x)^5 - \&c. \Big\};$$

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* *Crelle's Journal*, Vol. xv. p. 1.

and therefore by a known trigonometrical formula

$$\frac{d^{r-1}(1-z^2)^{\frac{2r-1}{2}}}{dz^{r-1}} = (-)^{r-1}\, 3\,.\,5 \ldots (2r-1)\, \frac{\sin rx}{r}.$$

Now if u be a function which, with its differentials up to the $(r-1)^{\text{th}}$, vanishes at the limits, we have, on integrating r times by parts between the limits,

$$\int dz\, u\, \frac{d^r v}{dz^r} = (-)^r \int dz\, v\, \frac{d^r u}{dz^r}.$$

Put $u = (1-z^2)^{\frac{2r-1}{2}}$, $v = f(z)$, then

$$\int_{-1}^{+1} dz\, (1-z^2)^{\frac{2r-1}{2}} f^{(r)}(z) = (-)^r \int_{-1}^{+1} dz\, f(z)\, \frac{d^r (1-z^2)^{\frac{2r-1}{2}}}{dz^r};$$

and as $$\frac{d^r (1-z^2)^{\frac{2r-1}{2}}}{dz^r} = \frac{dx}{dz}\cdot\frac{d}{dx}\cdot\frac{d^{r-1}}{dz^{r-1}}(1-z^2)^{\frac{2r-1}{2}}$$

$$= (-)^{r-1}\frac{dx}{dz}\, 3.5\ldots(2r-1)\cos rx;$$

we have, putting $z = \cos x$ and $dz = -\sin x\, dx$,

$$\int_0^\pi dx\, f^{(r)}(\cos x).(\sin x)^{2r} = 1.3.5\ldots(2r-1)\int_0^\pi dx\, f(\cos x).\cos rx.$$

(26) I shall conclude this Chapter on Definite Integrals with some examples of their application to the solution of partial differential equations. This mode of expressing the integrals of such equations was introduced by Lapalce*, and has been much employed by later writers, particularly Poisson†, Fourier‡, Cauchy, and Brisson‖.

It is particularly applicable to linear equations of orders higher than the first with constant coefficients, and it is useful because the solutions are put into a shape which facilitates the determination of the arbitrary functions. The principle of the method is to transform an explicit function not expressible in finite terms into an ordinary function involved under a definite integral; but the mode of transformation must be determined in each particular case by the nature of the function to be transformed, as will be seen in the following examples.

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* *Mémoires de l'Académie*, 1779.

† *Mémoires de l'Institut.* 1818, and *Journal Polytech.* Cah. XII.

‡ *Journal Polytech.* Cah. XII. and XIV.

‖ *Théorie de la Chaleur.*

(*a*) The integral of

$$\frac{dz}{dt} = a^2 \frac{d^2 z}{dx^2} \text{ is } z = \epsilon^{a^2 t \frac{d^2}{dx^2}} f(x),$$

(see Chap. VI. Sect. 1. Ex. 4. of the *Integ. Calc.*), and our object is to transform the operative function into one involving only the first power of $\frac{d}{dx}$. Now

$$\int_{-\infty}^{+\infty} d\omega\, \epsilon^{-\omega^2} = \pi^{\frac{1}{2}}, \text{ and } \int_{-\infty}^{+\infty} d\omega\, \epsilon^{-(\omega - b)^2} = \pi^{\frac{1}{2}}.$$

Therefore, putting $a \frac{d}{dx} t^{\frac{1}{2}}$ for b, and multiplying the two sides of the integral by the two sides of this equation,

$$\pi^{\frac{1}{2}} z = \int_{-\infty}^{+\infty} d\omega\, \epsilon^{-\left(\omega - a t^{\frac{1}{2}} \frac{d}{dx}\right)^2} \epsilon^{a^2 t \frac{d^2}{dx^2}} f(x)$$

$$= \int_{-\infty}^{+\infty} d\omega\, \epsilon^{-\omega^2} . \epsilon^{2 \omega a t^{\frac{1}{2}} \frac{d}{dx}} f(x);$$

therefore $\pi^{\frac{1}{2}} z = \int_{-\infty}^{+\infty} d\omega\, \epsilon^{-\omega^2} f(x + 2 \omega a t^{\frac{1}{2}})$.

This transformation is due to Laplace, *Jour. Polytech.* Cah. XV.

(*b*) The integral of

$$\frac{d^2 v}{dt^2} = a^2 \left(\frac{d^2 v}{dx^2} + \frac{d^2 v}{dy^2} + \frac{d^2 v}{dz^2} \right)$$

is, if we put $\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2} = D^2$,

$$v = \epsilon^{atD} F(x, y, z) + \epsilon^{-atD} f(x, y, z);$$

which may also be put under the form

$$2v = (\epsilon^{atD} - \epsilon^{-atD}) \phi(x, y, z) + (\epsilon^{atD} + \epsilon^{-atD}) \psi(x, y, z),$$

if $\phi + \psi = F$, $\phi - \psi = f$.

$$\epsilon^{atD} - \epsilon^{-atD} = aDt \int_0^\pi d\theta \sin\theta\, \epsilon^{atD\cos\theta};$$

nction ϕ is arbitrary, we may write ϕ instead hat

$$) \phi(x, y, z) = \int_0^\pi d\theta \sin\theta\, \epsilon^{atD\cos\theta}\, t\phi(x, y, z).$$

Ex. (4, d)

i $\epsilon^{atD\cos\theta}$

$$du\, dv \sin u\, \epsilon^{at\left(\frac{d}{dx}\sin u \sin v + \frac{d}{dy}\sin u \cos v + \frac{d}{dz}\cos u\right)};$$

efore

$$2\pi\left(\epsilon^{atD} - \epsilon^{-atD}\right) \phi(x, y, z)$$

$$\int_0^{2\pi} du\, dv \sin u \,.\, t \,.\, \phi(x + at \sin u \sin \quad y + at \sin u \cos v,\ z + at \cos u).$$

Also as $\epsilon^{atD} + \epsilon^{-atD} = (aD)^{-1}\dfrac{d}{dt}\left(\epsilon^{atD} - \epsilon^{-atD}\right)$, we find

$$= \int_0^\pi \int_0^{2\pi} du\, dv \sin u \,.\, t \,.\, \phi(x + at \sin u \sin v,\ y + at \sin u \cos v,\ z + at \cos u)$$

$$+ \frac{d}{dt} \int_0^\pi \int_0^{2\pi} du\, dv \sin u \,.\, t \,.\, \psi(x + at \sin u \sin v,\ y + at \sin u \cos v,\ z + at \cos u).$$

This transformation is given by Poisson, *Mémoires de l'Institut*, 1818.

(c) The equation for determining the vibratory motion of a thin elastic lamina is

$$\frac{d^2 z}{dt^2} + b^2 \frac{d^4 z}{dx^4} = 0,$$

the integral of which is

$$z = \cos\left(bt\frac{d^2}{dx^2}\right) f(x) + \sin\left(bt\frac{d^2}{dx^2}\right) F(x).$$

Now $\int_{-\infty}^{+\infty} dy\, \epsilon^{-2ay} \cos y^2 = \pi^{\frac{1}{2}} \cos\left(a^2 + \dfrac{\pi}{4}\right);$

and therefore $\int_{-\infty}^{+\infty} dy\, \epsilon^{-2ay} \cos\left(\dfrac{\pi}{4} - y^2\right) = \pi^{\frac{1}{2}} \cos a^2.$

Hence putting $bt\dfrac{d^2}{dx^2}$ for a^2,

$$\cos\left(bt\frac{d^2}{dx^2}\right) = \frac{1}{\pi^{\frac{1}{2}}}\int_{-\infty}^{+\infty} dy\, \epsilon^{-2(bt)^{\frac{1}{2}} y \frac{d}{dx}} \cos\left(\frac{\pi}{4} - y^2\right).$$

Also $\sin\left(bt\frac{d^2}{dx^2}\right) = b\frac{d^2}{dx^2}\int dt \cos\left(bt\frac{d^2}{dx^2}\right)$.

Therefore as $F(x)$ is an arbitrary function, and as we may write $F(x)$ for $b\frac{d^2}{dx^2}F(x)$, we have

$$\pi^{\frac{1}{2}} z = \int dy \cos\left(\frac{\pi}{4} - y^2\right) f\left\{x - 2y(bt)^{\frac{1}{2}}\right\}$$

$$+ \int dt \int dy \cos\left(\frac{\pi}{4} - y^2\right) F\left\{x - 2y(bt)^{\frac{1}{2}}\right\}.$$

Poisson, *Ib.*

CHAPTER XII.

COMPARISON OF TRANSCENDENTS.

The integration of differential expressions frequently leads to forms which are not expressible by any finite combination of algebraic, circular, and logarithmic functions. Such integrals are called transcendents, and the study of their properties becomes of importance as affording the means of classifying and arranging them so as to reduce them to the smallest number of independent functions.

The class of transcendents which has been most studied consists of those called *elliptic*, from their being in certain cases capable of representation by elliptic arcs. They thus appear to be functions little more complicated than those which are represented by circular arcs, and to be naturally pointed out as the next subject of investigation. The properties of these functions which have been discovered, relating chiefly to sums and differences of connected transcendents are very numerous; but in the following pages I shall confine myself to elementary illustrations of some of the principal theorems, making use chiefly of those examples which admit of a geometrical interpretation.

All elliptic transcendents are included in the formula

$$\int \frac{P dx}{(a + bx + cx^2 + dx^3 + ex^4)^{\frac{1}{2}}};$$

which can always by proper substitutions be reduced to others contained in

$$\int \frac{A + B(\sin\phi)^2}{C + D(\sin\phi)^2} \frac{d\phi}{\{1 - e^2(\sin\phi)^2\}^{\frac{1}{2}}}.$$

The fundamental relation connecting transcendents of this class was discovered by Euler*, and may be thus expressed.

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* *Novi Comm. Petrop.* Tom. VI. and VII.

If we have the equation

$$\frac{d\phi}{\{1-e^2(\sin\phi)^2\}^{\frac{1}{2}}} \pm \frac{d\psi}{(1-e^2\sin^2\psi)^{\frac{1}{2}}} = 0,$$

and if $\int \frac{d\phi}{\{1-e^2(\sin\phi)^2\}^{\frac{1}{2}}}$ be represented by $F(\phi)$, then will

$$F(\phi) \pm F(\psi) = F(\mu), \ldots\ldots\ldots\ldots\ldots (a)$$

ϕ, ψ, and μ being connected by the equation

$$\cos\phi\cos\psi \mp \sin\phi\sin\psi\{1-e^2(\sin\mu)^2\}^{\frac{1}{2}} = \cos\mu \ldots (a').$$

From this last may be deduced the following:

$$\sin\mu = \frac{\sin\phi\cos\psi\,\Delta(\psi) \pm \sin\psi\cos\phi\,\Delta(\phi)}{1-e^2(\sin\phi)^2(\sin\psi)^2} \ldots (b')$$

$$\tan\mu = \frac{\tan\phi\,\Delta(\psi) \pm \tan\psi\,\Delta(\phi)}{1 \mp \tan\phi\tan\psi\,\Delta(\phi)\,\Delta(\psi)}, \ldots\ldots\ldots\ldots (c')$$

where $\Delta(\phi) = \{1-e^2(\sin\phi)^2\}^{\frac{1}{2}}$.

The functions F are called elliptic functions of the first kind; e is called the modulus, and ϕ the amplitude of the function. Hence it appears that we can determine algebraically the amplitude of a function which is equal to the sum or difference of two given functions, and therefore also of one which is a multiple or a part of a given function. Thus if ϕ_2 be the amplitude of a function which is double of that whose amplitude is ϕ, we have to put $\psi=\phi$ and $\mu=\phi_2$ in the preceding expressions, when we find as an equation for determining ϕ_2,

$$\tan\tfrac{1}{2}\phi_2 = \tan\phi\,\{1-e^2(\sin\phi)^2\}^{\frac{1}{2}}.$$

Ex. (1) The preceding theorems may be at once applied to demonstrate certain properties of arcs of the lemniscate which were at an early period discovered by Fagnani*.

The equation to the curve being

$$(x^2+y^2)^2 = a^2(x^2-y^2),$$

we have $$s = a^2\int\frac{dx\,(x^2+y^2)^{\frac{1}{2}}}{y\,\{2(x^2+y^2)+a^2\}};$$

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* *Produzioni Mathematiche*, Tom. II.

which, if we assume

$$x = a\cos\phi\,\{1 - \tfrac{1}{2}(\sin\phi)^2\}^{\frac{1}{2}}, \quad y = \frac{a}{2^{\frac{1}{2}}}\sin\phi\cos\phi$$

becomes
$$s = \frac{a}{2^{\frac{1}{2}}}\int\frac{d\phi}{\{1 - \frac{1}{2}(\sin\phi)^2\}^{\frac{1}{2}}}.$$

Thus the arc of the lemniscate is at once expressed by the transcendent F, so that we write

$$s = \frac{a}{2^{\frac{1}{2}}}F(\phi),$$

the modulus being $(\frac{1}{2})^{\frac{1}{2}}$; and the theorems proved of $F(\phi)$ may be interpreted in the case of arcs of the lemniscate. Let s_1, s_2, s_3 be the arcs corresponding to the amplitudes ϕ, ψ, μ, then

$$s_1 + s_2 = s_3,$$

provided that ϕ, ψ, and μ satisfy the equation

$$\cos\phi\cos\psi - \sin\phi\sin\psi\,\{1 - \tfrac{1}{2}(\sin\mu)^2\}^{\frac{1}{2}} = \cos\mu.$$

If $\mu = \frac{1}{2}\pi$ the corresponding arc is the quadrant of the lemniscate, and if $\psi = \phi$, the value of ϕ given by the equation

$$(\sin\phi)^2 = \frac{2^{\frac{1}{2}}}{1 + 2^{\frac{1}{2}}}$$

will give the amplitude corresponding to the middle point of the quadrant.

Functions of the form

$$\int d\phi\,\{1 - e^2(\sin\phi)^2\} = E(\phi)$$

are called elliptic functions of the second kind, and are connected by the following equation. The same relation between the amplitudes which satisfies the equation

$$F(\phi) + F(\psi) - F(\mu) = 0$$

will also satisfy the equation

$$E(\phi) + E(\psi) - E(\mu) = e^2\sin\phi\sin\psi\sin\mu \ldots\ldots\ldots\ldots (b).$$

(2) This formula leads at once to the demonstration of several properties of elliptic arcs: for if s be the arc cor-

responding to the abscissa x in an ellipse whose major-axis is 1 and eccentricity is e,

$$s = \int dx \frac{(1 - e^2x^2)^{\frac{1}{2}}}{(1 - x^2)^{\frac{1}{2}}}.$$

If we assume $x = \cos\phi$ this becomes

$$s = \int d\phi \left\{1 - e^2 (\sin\phi)^2\right\}^{\frac{1}{2}} = E(\phi).$$

Now let $\mu = \frac{1}{2}\pi$, then $E(\frac{1}{2}\pi)$ represents the quadrantal arc of the ellipse, and equation (b') gives as the relation between ϕ and ψ

$$(1 - e^2)^{\frac{1}{2}} \tan\phi \tan\psi = 1;$$

whence $e^2 \sin\phi \sin\psi = \dfrac{e^2 \sin\phi \cos\phi}{\left\{1 - e^2(\sin\phi)^2\right\}^{\frac{1}{2}}} = \dfrac{e^2 \sin\psi \cos\psi}{\left\{1 - e^2(\sin\psi)^2\right\}^{\frac{1}{2}}}$.

Now in (fig. 63) take $BM = E(\phi)$, $BN = E(\psi)$, then will

$$AN = E(\tfrac{1}{2}\pi) - E(\psi),$$

$$\text{and} \quad e^2 \sin\phi \sin\psi = MY = NZ,$$

or the portion of the tangent intercepted between the point of contact and the perpendicular on it from the centre. From equation (b) then it appears that

$$BM - AN = MY = NZ;$$

or the difference between two arcs of an ellipse is equal to an assignable straight line. This remarkable Theorem was discovered by Fagnani*.

(3) If $\psi = \phi$, in which case

$$\tan^2\phi = \frac{1}{(1 - e^2)^{\frac{1}{2}}},$$

the two points M and N coincide in K, and we have

$$BK - AK = 1 - (1 - e^2)^{\frac{1}{2}},$$

or the difference of the two arcs is equal to the difference of the two axes of the ellipse.

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* *Produzioni Mathematiche*, Tom. II.

(4) Again, let us suppose $\psi = \phi$, while μ retains a general value: then we have

$$2E(\phi) - E(\mu) = e^2 (\sin\phi)^2 \sin\mu;$$

while equation (a') gives

$$(\cos\phi)^2 - (\sin\phi)^2 \{1 - e^2 (\sin\mu)^2\}^{\frac{1}{2}} = \cos\mu,$$

the condition for doubling or bisecting elliptic arcs. For doubling, we have

$$\sin\mu = \frac{2\sin\phi\cos\phi\,\Delta(\phi)}{1 - e^2(\sin\phi)^4}, \text{ or } \tan\tfrac{1}{2}\mu = \Delta(\phi)\tan\phi;$$

and for bisecting

$$(\sin\phi)^2 = \frac{1 - \cos\mu}{1 + \Delta(\mu)}.$$

From this it appears that we can always determine an arc of an ellipse which shall differ from the double or the half of another by an algebraical quantity. Under certain circumstances, however, when the origin of the arc is arbitrary, this difference may be made to disappear, as in the following problem.

(5) Find an arc MN which shall be exactly half of the quadrant BA of the ellipse.

Let ϕ be the amplitude of the point M, ψ that of N, θ that of K, determined as before by the condition

$$\tan^2\theta = \frac{1}{(1 - e^2)^{\frac{1}{2}}},$$

$$\text{so that } E(\theta) = \tfrac{1}{2}E(\tfrac{1}{2}\pi) + \tfrac{1}{2}\{1 - (1 - e^2)^{\frac{1}{2}}\}.$$

Now if we have

$$F(\phi) + F(\theta) - F(\psi) = 0,$$

it follows that

$$E(\phi) + E(\theta) - E(\psi) = e^2 \sin\phi\sin\psi\sin\theta.$$

Therefore

$$E(\psi) - E(\phi) = \tfrac{1}{2}E(\tfrac{1}{2}\pi) + \tfrac{1}{2}\{1 - (1 - e^2)^{\frac{1}{2}}\} - e^2\sin\phi\sin\psi\sin\theta.$$

Hence, in order that MN or $E(\psi) - E(\phi)$ may be equal to $\frac{1}{2}E(\frac{1}{2}\pi)$, we must have

$$1 - (1 - e^2)^{\frac{1}{2}} = 2e^2 \sin\phi \sin\psi \sin\theta,$$

which gives $2\sin\phi\sin\psi = \sin\theta$.

On the other hand, we have from (a')

$$2\cos\phi\cos\psi = \cos\theta;$$

from which equations ϕ and ψ may be found, and so the arc MN determined.

Abel, in the third volume of *Crelle's Journal*, has given a remarkable theorem for finding the sums of transcendental integrals, the amplitudes of which are connected by certain conditions. The following is the enunciation of the theorem.

Let $\phi(x)$ be an integral function of x which is decomposable into two integral factors, so that

$$\phi(x) = \phi_1(x) \,.\, \phi_2(x):$$

Let $f(x)$ be another integral function of x, and let

$$\psi(x) = \int \frac{f(x)\,dx}{(x-a)\{\phi(x)\}^{\frac{1}{2}}},$$

where a is any constant quantity: also let

$$\theta_1(x) = a_0 + a_1x + a_2x^2 + \&c. + a_nx^n,$$

$$\theta_2(x) = c_0 + c_1x + c_2x^2 + \&c. + c_mx^m,$$

where a_0, a_1, a_2, &c., c_0, c_1, c_2, &c. are any quantities whatsoever of which one at least is variable. Then if

$$\{\theta_1(x)\}^2\phi_1(x) - \{\theta_2(x)\}^2\phi_2(x) = A(x-x_1)(x-x_2)\ldots(x-x_\mu),$$

where A is independent of x, we shall have,

$$\epsilon_1\psi(x_1) + \epsilon_2\psi(x_2) + \epsilon_3\psi(x_3) \ldots + \epsilon_\mu\psi(x_\mu)$$

$$= \frac{f(a)}{\{\phi(a)\}^{\frac{1}{2}}} \log\left[\frac{\theta_1(a)\{\phi_1(a)\}^{\frac{1}{2}} + \theta_2(a)\{\phi_2(a)\}^{\frac{1}{2}}}{\theta_1(a)\{\phi_1(a)\}^{\frac{1}{2}} - \theta_2(a)\{\phi_2(a)\}^{\frac{1}{2}}}\right] + r + C,$$

where C is a constant, and r is the coefficient of $\frac{1}{x}$ in the development of the function

$$\frac{f(x)}{(x-a)\{\phi(x)\}^{\frac{1}{2}}}\log\left[\frac{\theta_1(x)\{\phi_1(x)\}^{\frac{1}{2}}+\theta_2(x)\{\phi_2(x)\}^{\frac{1}{2}}}{\theta_1(x)\{\phi_1(x)\}^{\frac{1}{2}}-\theta_2(x)\{\phi_2(x)\}^{\frac{1}{2}}}\right],$$

according to descending powers of x.

The quantities ϵ_1, $\epsilon_2 \ldots \epsilon_\mu$, are equal to $+1$ or to -1, and depend on the values of x_1, $x_2 \ldots x_\mu$, being determined by the equation

$$\theta_1(x)\{\phi_1(x)\}^{\frac{1}{2}} = \epsilon\theta_2(x)\{\phi_2(x)\}^{\frac{1}{2}}.$$

(6) Let the integral which is to be considered be

$$\int\frac{dx}{(1+x^n)^{\frac{1}{2}}}.$$

In this case assume

$$1+x^n=\phi(x),\quad 1+x^n=\phi_1(x),\quad 1=\phi_2(x),$$

$$f(x)=x-a,\quad \theta_1(x)=1,\quad \theta_2(x)=1+c_1x:\ \text{then}$$

$$\{\theta_1(x)\}^2.\phi_1(x)-\{\theta_2(x)\}^2\phi_2(x)=1+x^n-(1+c_1x)^2$$
$$=x(x^{n-1}-c_1^2x-2c_1).$$

The roots of this being 0, x_1, $x_2 \ldots x_{n-1}$, we have $n-2$ relations between them from the conditions implied in the equation

$$x^{n-1}-c_1^2x-2c_1=0:$$

these are

$$\Sigma(x_1)=0,\ \Sigma(x_1x_2)=0 \ldots \Sigma\left(\frac{1}{x_1}\right)=\pm\frac{x_1x_2\ldots x_{n-1}}{4},$$

the upper or lower sign of the last equation being taken according as n is odd or even.

Since $f(x)=x-a$, $f(a)=0$,

and as $f(x)$ is of lower dimensions than $\{\phi(x)\}^{\frac{1}{2}}$ we have $r=0$; consequently

$$\epsilon_1\int_0^{x_1}\frac{dx}{(1+x^n)^{\frac{1}{2}}}+\epsilon_2\int_0^{x_2}\frac{dx}{(1+x^n)^{\frac{1}{2}}}+\ldots+\epsilon_{n-1}\int_0^{x_{n-1}}\frac{dx}{(1+x^n)^{\frac{1}{2}}}=\text{const.}$$

Lubbock, *Phil. Mag. New Series*, Vol. VI. p. 121.

Fagnani has availed himself of the relation which subsists between the integrals

$$\int y\,dx \text{ and } \int x\,dy,$$

to compare certain transcendents of considerable interest. Since

$$\int x\,dy + \int y\,dx = xy + \text{const.},$$

if a symmetrical equation subsist between x and y, so that x is the same function of y that y is of x, or that when

$$x = \phi(y), \quad y = \phi(x);$$

it follows that

$$\int \phi(x)\,dx + \int \phi(y)\,dy = xy + \text{const.}$$

This is true whatever be the nature of ϕ, independently of the integrability of the functions.

(7) Thus if x be the abscissa of a hyperbola, the major axis of which is unity, the corresponding arc is represented by

$$\int dx \left(\frac{e^2x^2 - 1}{x^2 - 1}\right)^{\frac{1}{2}}$$

(e being the excentricity.)

If y be another abscissa connected with the former by the equation

$$ey = \left(\frac{e^2x^2 - 1}{x^2 - 1}\right)^{\frac{1}{2}},$$

or $$e^2x^2y^2 - e^2(x^2 + y^2) + 1 = 0,$$

which is symmetrical with respect to x and y, it follows that

$$ex = \left(\frac{e^2y^2 - 1}{y^2 - 1}\right)^{\frac{1}{2}}.$$

Therefore

$$\int dx \left(\frac{e^2x^2 - 1}{x^2 - 1}\right)^{\frac{1}{2}} + \int dy \left(\frac{e^2y^2 - 1}{y^2 - 1}\right)^{\frac{1}{2}} = exy + \text{const.}$$

Mr Fox Talbot* has extended to any number of variables the principle made use of by Fagnani in the case of two, and he has arrived at the following Theorem.

If there be n variables x, y, z, &c. connected by $(n - 1)$ *symmetrical* equations, so that they are all similar functions of each other, then if

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* *Phil. Trans.* 1836 and 1837.

$$\frac{xyz\ldots}{x} = \phi(x), \quad \frac{xyz\ldots}{y} = \phi(y), \text{ \&c.}$$

we shall have

$$\int\phi(x)\,dx + \int\phi(y)\,dy + \int\phi(z)\,dz + \text{\&c.} = xyz \text{ \&c.} + \text{const.}$$

(8) Let the variables be three in number, and let the two symmetrical conditions be

$$x + y + z = 0,$$
$$(x^2 - 1)(y^2 - 1)(z^2 - 1) + 1 = 0.$$

Then $yz = \phi(x) = \left(\dfrac{1 + x^2 - x^4}{1 - x^2}\right)^{\frac{1}{2}} - 1,$

and similarly for the other variables. Hence, by the theorem,

$$\int dx\left(\frac{1+x^2-x^4}{1-x^2}\right)^{\frac{1}{2}} + \int dy\left(\frac{1+y^2-y^4}{1-y^2}\right)^{\frac{1}{2}} + \int dz\left(\frac{1+z^2-z^4}{1-z^2}\right)^{\frac{1}{2}} = xyz + C;$$

since $x + y + z = 0$, by the first condition.

(9) Let three variables be connected by the equations,

$$x + y + z = 0,$$
$$xy + xz + yz + \frac{x^2y^2z^2}{4} = 0.$$

In this case

$$\int yz\,dx = \int\phi(x)\,dx = 2\int dx\,\frac{(1 + x^4)^{\frac{1}{2}}}{x^2} + \frac{2}{x}.$$

But $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = -\dfrac{xyz}{4}$; therefore

$$\int dx\,\frac{(1 + x^4)^{\frac{1}{2}}}{x^2} + \int dy\,\frac{(1 + y^4)^{\frac{1}{2}}}{y^2} + \int dz\,\frac{(1 + z^4)^{\frac{1}{2}}}{z^2} = \frac{3}{4}xyz + C.$$

Now $\int dx\,\dfrac{(1 + x^4)^{\frac{1}{2}}}{x^2}$ is the arc of an equilateral hyperbola referred to its asymptotes, and it therefore appears that three hyperbolic arcs may be determined in an infinite number of ways, so that their sum may be an algebraic quantity. It is to be observed that the two symmetrical equations connecting x, y, and z, are equivalent to saying that these quantities are the roots of the cubic equation

$$x^3 - \frac{r^2}{4} x - r = 0.$$

(10) Let there be three variables connected by the equations

$$x^3 + y^3 + z^3 = 0,$$

$$x^3y^3 + x^3z^3 + y^3z^3 + 1 = 0;$$

then $yz = \phi(x) = (x^6 - 1)^{\frac{1}{3}}$, and similarly for the others.

Hence it follows that

$$\int dx\,(x^6 - 1)^{\frac{1}{3}} + \int dy\,(y^6 - 1)^{\frac{1}{3}} + \int dz\,(z^6 - 1)^{\frac{1}{3}} = xyz + C.$$

This process is deficient in generality since it supposes that the sum of the integrals is equal to an algebraical function of one form only, viz. the product of the variables; and moreover it is limited in its application, as no means are given by which we can determine what are the symmetrical relations which must exist between the variables, in order that the sum of the integrals of a given function may have an algebraical value. Mr Fox Talbot, however, has given an exposition of a more general method, which is free from these objections, and of this I subjoin a sketch with some applications.

It is easy to see that the supposition that $(n-1)$ symmetrical relations exist between the n variables is equivalent to saying that the n variables are roots of an equation of n dimensions, one of whose coefficients at least is variable, the others being either constant or functions of the variable one. Let therefore x, y, z, &c. be roots of an equation

$$x^n - p_1x^{n-1} + p_2x^{n-2} + \&\text{c.} \pm p_n = 0;$$

then not only are the coefficients p_1, $p_2 \ldots p_n$ symmetrical functions of the roots, but so likewise are all combinations of these coefficients. Let v be a general symbol representing any one of the coefficients, or any combination of them; then v may be considered either as a function of all the roots, or only of any one of them, in which latter case the root may be changed for any other without altering the value of v. Such a function Mr Talbot calls a "symmetrical" of the equation, and this name we shall employ. Now if v be a symmetrical of any equation

$$v = f(x) = f(y) = f(z) = \&c.;$$

and if we denote the sum of the differentials of the roots by $S.dx$, we shall have

$$f(x).S.dx = S.f(x)\,dx. \ldots\ldots\ldots\ldots (1).$$

More generally, if

$$S.\psi(x)\,dx = \psi(x)\,dx + \psi(y)\,dy + \psi(z)\,dz + \&c.$$

$$f(x).S.\psi(x)\,dx = S.f(x).\psi(x)\,dx, \ldots\ldots (2)$$

provided that $f(x)$ be a symmetrical of the equation the roots of which are x, y, z, &c.

Now if $\int X dx$ be the required integral, X being a function of x, and we wish to determine the number of variables and the equation of which they must be the roots in order that

$$S.\int X dx$$

may have an algebraical value, we must assume

$$X = v, \text{ (} v \text{ being a symmetrical)},$$

and if this equation can be cleared of radicals, it may be reduced to the form

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \&c. \pm p_n = 0.$$

In this the index n determines the number of the variables, and the equation found is that of which X is a symmetrical. Now

$$x + y + z + \&c. = p_1 = \phi(v) \text{ suppose};$$

therefore $$S.dx = dp_1 = d.\phi(v),$$

and $$XS.dx = S.Xdx = vd.\phi(v).$$

Hence $$\int S.Xdx = S.\int Xdx = \int vd.\phi(v) \ldots\ldots (3).$$

In order that this may be algebraical, it is only necessary that $\int\phi(v)\,dv$ should be capable of integration. If $\phi(v) =$ const., or $d.\phi(v) = 0$,

$$S.\int X\,dx = \text{const.} \dots\dots\dots\dots\dots\dots (4).$$

Instead of assuming $X = v$ we might assume $X = v.\psi(x)$, or $X = v + \psi(x)$, or generally

$$X = f(v, x),$$

according as each assumption may appear to simplify the case under consideration. It is to be observed that if X contain only one power of the variable x, it cannot be assumed to be a symmetrical of any equation, as that would lead to the conclusion that

$$x = y = z = \dots\dots,$$

whereas it is supposed that the roots are in general all different. Hence in this case we cannot assume $X = v$, and we must make use of some of the other assumptions.

In the same way if we wish to find an algebraical relation between n variables x, y, z, such that

$$S.\int\phi(X)\,dx = \text{const.},$$

X being of the form $x^n - a_1x^{n-1} + a_2x^{n-2} + \text{\&c.} + k$, all the coefficients being constants, and ϕ being any function whatever, we assume $X = v$, and therefore

$$x^n - a_1x^{n-1} + a_2x^{n-2} + \dots + (k - v) = 0.$$

This equation has only one variable coefficient $(k - v)$, and as $x + y + z + \text{\&c.} = a_1$ is constant

$$S.dx = 0.$$

Now X being a symmetrical $\phi(X)$ is one also: therefore

$$\phi(X).S\,dx = 0 = S.\phi(X)\,dx,$$

and $$S.\int\phi(X)\,dx = \text{const.}$$

(11) Let the function be $\left(\dfrac{1 + x^2}{x}\right)^{\frac{1}{2}}$, and assume

$$\frac{1 + x^2}{x} = v, \quad \text{or} \quad x^2 - vx + 1 = 0.$$

Then $$S \,.\, dx = dv,$$

and $$S \,.\, \left(\frac{1+x^2}{x}\right)^{\frac{1}{2}} dx = v^{\frac{1}{2}} dv;$$

therefore $$S \,.\, \int dx \left(\frac{1+x^2}{x}\right)^{\frac{1}{2}} = \tfrac{2}{3} v^{\frac{3}{2}} + C;$$

which is equivalent to saying that

$$\int dx \left(\frac{1+x^2}{x}\right)^{\frac{1}{2}} + \int dy \left(\frac{1+y^2}{y}\right)^{\frac{1}{2}} = \tfrac{2}{3}(x+y)^{\frac{3}{2}} + C,$$

provided that $xy = 1$: for x and y being the roots of $x^2 - vx + 1 = 0$, we have

$$x + y = v \quad \text{and} \quad xy = 1.$$

(12) Let the proposed integral be $\int \frac{dx}{(1-x^3)^{\frac{1}{2}}}$.

Now $(1-x^3)^{\frac{1}{2}} = (1-x)\left(\frac{1+x+x^2}{1-x}\right)^{\frac{1}{2}}$; therefore assume $\frac{1+x+x^2}{1-x} = v$, or

$$x^2 + (1+v)\,x + (1-v) = 0.$$

This being an equation of the second degree, the number of variables is two, and therefore

$$\int \frac{dx}{(1-x^3)^{\frac{1}{2}}} + \int \frac{dy}{(1-y^3)^{\frac{1}{2}}} = \int \frac{dx}{v^{\frac{1}{2}}(1-x)} + \int \frac{dy}{v^{\frac{1}{2}}(1-y)}.$$

But from the equation of condition, we have

$$x + y = -(1+v), \quad \text{and} \quad xy = 1 - v;$$

whence by eliminating v, we obtain

$$xy - (x+y) = 2,$$

as the equation of condition connecting x and y.

From this we deduce

$$(1-x)(1-y) = 3,$$

and therefore $\frac{dx}{1-x} + \frac{dy}{1-y} = 0$. Hence

$$\int \frac{dx}{(1-x^3)^{\frac{1}{2}}} + \int \frac{dy}{(1-y^3)^{\frac{1}{2}}} = \text{const.}$$

(13) If we wish to find the condition that the sum of three integrals of this form may be constant, we write

$$\int\frac{dx}{(1-x^3)^{\frac{1}{2}}}=\int\frac{dx}{x}\left(\frac{x^2}{1-x^3}\right)^{\frac{1}{2}} \text{ and assume}$$

$$\frac{x^2}{1-x^3}=\frac{1}{v}, \text{ or } x^3+vx^2-1=0.$$

Hence the three variables x, y, z are connected by the equations

$$xyz=1, \quad xz+yz+xy=0.$$

From the first of these we have

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0;$$

multiply by $\frac{1}{v^{\frac{1}{2}}}$ and integrate, when we have

$$\int\frac{dx}{(1-x^3)^{\frac{1}{2}}}+\int\frac{dy}{(1-y^3)^{\frac{1}{2}}}+\int\frac{dz}{(1-z^3)^{\frac{1}{2}}}=\text{const.}$$

(14) Let the integral be $\int\frac{dx}{(x^3-1)^{\frac{1}{3}}}$.

Assume $x^3=t$ when the integral becomes

$$\tfrac{1}{3}\int\frac{dt}{(t^3-t^2)^{\frac{1}{3}}}.$$

Put $t^3-t^2=v$, or $t^3-t^2-v=0$, then

$$\int\frac{dx}{(x^3-1)^{\frac{1}{3}}}+\int\frac{dy}{(y^3-1)^{\frac{1}{3}}}+\int\frac{dz}{(z^3-1)^{\frac{1}{3}}}=\text{const.},$$

provided that the three variables satisfy the equations

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=1, \quad \text{and } x^{\frac{1}{3}}z^{\frac{1}{3}}+y^{\frac{1}{3}}z^{\frac{1}{3}}+x^{\frac{1}{3}}y^{\frac{1}{3}}=0.$$

By the same method various curious properties may be demonstrated of circular, parabolic, and elliptic arcs.

(15) If the sines of three circular arcs be the roots of the equation

$$x^3+rx^2+\left(\frac{r^2}{4}-1\right)x-r=0,$$

the sum of the arcs is constant. The word sum is to be understood in the algebraic sense as including subtraction, when one or more of the arcs is taken negatively.

The radius of the circle being 1, and the sine of the arc being x, the arc itself is equal to

$$\int \frac{dx}{(1-x^2)^{\frac{1}{2}}}.$$

Assume $\dfrac{1}{(1-x^2)^{\frac{1}{2}}} = vx + 1$, whence

$$x^3 + \frac{2}{v}x^2 + \left(\frac{1}{v^2} - 1\right)x - \frac{2}{v} = 0.$$

In this case the symmetrical $v = \dfrac{2}{r}$, where r is the product of the roots. Hence the equation may be written

$$x^3 + rx^2 + \left(\frac{r^2}{4} - 1\right)x - r = 0.$$

Also since $\dfrac{1}{(1-x^2)^{\frac{1}{2}}} = vx + 1$,

$$S.\frac{dx}{(1-x^2)^{\frac{1}{2}}} = vS.xdx + Sdx.$$

Now $S.x^2 = \dfrac{r^2}{2} + 2$, therefore $S.xdx = \dfrac{rdr}{2}$;

also $Sdx = -dr$, therefore

$$S.\frac{dx}{(1-x^2)^{\frac{1}{2}}} = dr - dr = 0, \text{ and}$$

$$S.\int\frac{dx}{(1-x^2)^{\frac{1}{2}}} = \text{const.},$$

or the sum of the arcs is constant.

(16) If the equation to the parabola be

$$2y = x^2,$$

and if three abscissæ be the roots of the equation

$$x^3 - rx^2 + \left(\frac{r^2}{4} + 1\right)x - r = 0,$$

the sum of the arcs is equal to the sum of the abscissæ.

The expression for the arc is $\int dx\,(1+x^2)^{\frac{1}{2}}$; assume then $(1+x^2)^{\frac{1}{2}} = x^2 + vx + 1$, which leads to the equation

$$x^3 - rx^2 + \left(\frac{r^2}{4} + 1\right)x - r = 0,$$

in which $r = -2v$. Now

$$S \,.\, dx\,(1+x^2)^{\frac{1}{2}} = S \,.\, x^2\,dx + v\,S \,.\, x\,dx + S\,dx.$$

But from the equation of condition

$$S \,.\, x^3 = \frac{r^3}{4}, \quad S \,.\, x^2 = \frac{r^2}{2} - 2,$$

therefore $S \,.\, x^2 dx + vS \,.\, x\,dx = 0$, and

$$S \,.\, dx\,(1+x^2)^{\frac{1}{2}} = S\,dx,$$

so that $S \,.\, \int dx\,(1+x^2)^{\frac{1}{2}} = S \,.\, x + C.$

It appears on trial that $C = 0$, and consequently the sum of the arcs is equal to the sum of the abscissæ or to r.

(17) If three abscissæ of an ellipse be roots of the equation

$$x^3 + rx^2 + \left(\frac{1-e^2}{4}r^2 - 1\right) - r = 0,$$

the sum of the arcs is $C \pm e^2 r$.

This is an extension of the theorem in (15) and may be proved like it by assuming

$$\left(\frac{1 - e^2x^2}{1 - x^2}\right)^{\frac{1}{2}} = vx + 1.$$

The works in which the reader will find full developments of the Theory of Elliptic Transcendents are Legendre, *Exercices de Calcul Integral*, and *Théorie des Fonctions Elliptiques*; Jacobi, *Nova Theoria Func. Ellip.*; Abel, *Œuvres.* Besides these there are a number of memoirs on this subject in the different Journals and Transactions, such as those of Poisson, Crelle's *Journal*, Vol. XII; of Gudermann in the same collection, and Ivory, *Phil. Trans.* 1831.

Spence, in his *Mathematical Essays*, has given the name of *Logarithmic Transcendents* to functions of which the general form is

$$\pm x - \frac{x^2}{2^n} \pm \frac{x^3}{3^n} - \frac{x^4}{4^n} \pm \frac{x^5}{5^n} - \&c.$$

which he denotes by the characteristic symbol

$$L_n(1 \pm x).$$

It is easily seen that when $n = 1$ the series is that of the logarithm of $1 \pm x$, according as the upper or lower sign is taken, so that

$$L_1(1 \pm x) = \log(1 \pm x).$$

All these transcendents, including the logarithm, may be expressed by means of integrals which have a mutual dependence on each other. Thus

$$L_1(1 \pm x) = \log(1 \pm x) = \int \frac{\pm\, dx}{1 \pm x},$$

$$L_2(1 \pm x) = \int \frac{dx}{x} L_1(1 \pm x),$$

$$L_3(1 \pm x) = \int \frac{dx}{x} L_2(1 \pm x),$$

....

$$L_n(1 \pm x) = \int \frac{dx}{x} L_{n-1}(1 \pm x).$$

From these integrals various properties of the transcendents may be deduced by analytical transformations, some of which are here given.

(18) Omitting $L_1(1 \pm x)$, as it is a transcendent the properties of which are well known, let us take

$$L_2(1 \pm x) = \int \frac{dx}{x} \log(1 \pm x).$$

Using the lower sign, and changing $1 - x$ into x, and x into $1 - x$, we have

$$L_2(x) = \int \frac{-\,dx}{1 - x} \log(x).$$

Adding this to the equation

$$L_2(1-x) = \int \frac{dx}{x} \log(1-x),$$

we have

$$L_2(x) + L_2(1-x) = \int dx \left\{ \frac{\log(1-x)}{x} - \frac{\log x}{1-x} \right\}$$

$$= \log(x) . \log(1-x) + C.$$

To determine the constant, make $x = 0$, when as $\log(1) = 0$ and $L_2(1) = 0$, we have

$C = L_2(0) = -\frac{\pi^2}{6}$ by a known theorem. Hence

$$L_2(x) + L_2(1-x) = \log(x) . \log(1-x) - \frac{\pi^2}{6}.$$

This property of the transcendent L_2 is only true so long as x is less than unity, as when any greater value is assigned to it $\log(1-x)$ becomes impossible.

Euler, *Commen. Petrop.* 1738.

(19) Again in the equation

$$L_2(1-x) = \int \frac{dx}{x} \log(1-x),$$

we have by changing x into x^2,

$$L_2(1-x^2) = 2\int \frac{dx}{x} \log(1-x^2)$$

$$= 2\int \frac{dx}{x} \log(1+x) + 2\int \frac{dx}{x} \log(1-x).$$

Hence $L_2(1-x^2) = 2\,L_2(1+x) + 2\,L_2(1-x)$.

(20) If we take the upper sign, and in

$$L_2(1+x) = \int \frac{dx}{x} \log(1+x)$$

change x into $\frac{1}{x}$, it becomes

$$L_2\left(\frac{1+x}{x}\right) = -\int \frac{dx}{x} \log\left(\frac{1+x}{x}\right) = -\int \frac{dx}{x} \left\{\log(1+x) - \log x\right\}.$$

Now $$\int \frac{dx}{x} \log x = \tfrac{1}{2}(\log x)^2;$$

therefore, integrating,

$$L_2\left(\frac{1+x}{x}\right) + L_2(1+x) = \tfrac{1}{2}(\log x)^2 + C.$$

By putting $x = 1$ we find $C = 2\,L_2(2)$.

But in the equation

$$L_2(1-x^2) = 2\,L_2(1+x) + 2\,L_2(1-x),$$

if we put $x = 1$ we find

$$2\,L_2(2) = -L_2(0) = \frac{\pi^2}{6};$$

therefore $$L_2\left(\frac{1+x}{x}\right) + L_2(1+x) = \tfrac{1}{2}(\log x)^2 + \frac{\pi^2}{6}.$$

Analogous properties may by the same method be demonstrated of

$$L_3(1 \pm x) = \int \frac{dx}{x} L_2(1 \pm x),$$

and $$L_4(1 \pm x) = \int \frac{dx}{x} L_3(1 \pm x),$$

and so on in succession. Generally, the student will have no difficulty in demonstrating the following propositions:

$$L_n(1-x^2) = 2^{n-1} L_n(1+x) + 2^{n-1} L_n(1-x),$$

$$L_{2n}(1+x) + L_{2n}\left(\frac{1+x}{x}\right) = 2\,L_{2n}(2) + 2\,L_{2n-2}(2)\frac{(\log x)^2}{1\,.\,2} + \&\text{c.} + \frac{(\log x)^{2n}}{1\,.\,2\,.\,3\,.\,4 \ldots 2n},$$

$$L_{2n-1}(1+x) - L_{2n-1}\left(\frac{1+x}{x}\right) = 2\,L_{2n-2}(2)\log x + 2\,L_{2n-4}(2)\frac{(\log x)^3}{1\,.\,2\,.\,3} + \&\text{c.} + \frac{(\log x)^{2n-1}}{1\,.\,2\,.\,3\,.\,4 \ldots (2n-1)}.$$

Spence has extended this analysis to the investigation of the properties of transcendents defined by the general law

$$\phi_n(x) = \int \frac{dx}{x} \phi_{n-1}(x);$$

the final function or $\phi_0(x)$ being such that it remains unchanged when $\frac{1}{x}$ is substituted for x, or

$$\phi_0(x) = \phi_0 \left(\frac{1}{x}\right).$$

But for this investigation and others connected with it the reader is referred to the work before quoted.

The transcendents which we have been considering are all such that they may be derived by direct integration from known functions, but there are many other transcendents which are given only by means of differential equations. As these are frequently functions of great utility in physical researches, the study of their properties without integrating the equations in which they are involved becomes of great importance. Two examples of such investigations are subjoined.

(21) Let V be a function of x and r given by the differential equation of the second order

$$\frac{d}{dx}\left(k \frac{dV}{dx}\right) + (gr - l)\, V = 0, \dots\dots\dots\dots (1)$$

in which g, k, and l are functions of x, and r is a variable parameter; and if V also satisfy the conditions

$$\frac{dV}{dx} - h_1 V = 0 \text{ when } x = x_1 \dots\dots\dots\dots (2)$$

$$\frac{dV}{dx} + h_2 V = 0 \text{ when } x = x_2, \dots\dots\dots\dots (3)$$

then will

$$\int_{x_1}^{x_2} g\, dx\, V_m V_n = 0;$$

V_m and V_n being values of V corresponding to the values r_m and r_n of r.

From the given equation (1) we easily obtain

$$(r_m - r_n) g V_m V_n = V_m \frac{d}{dx}\left(k \frac{dV_n}{dx}\right) - V_n \frac{d}{dx}\left(k \frac{dV_m}{dx}\right);$$

therefore integrating with respect to x,

$$(r_m - r_n) \int dx g V_m V_n = k \left(V_m \frac{dV_n}{dx} - V_n \frac{dV_m}{dx}\right).$$

But from the condition (2) we find on taking the limit $x = x_1$, that

$$V_m \frac{dV_n}{dx} - V_n \frac{dV_m}{dx} = 0.$$

Similarly we find from (3) that at the limit $x_1 = x_2$ the same relation holds; hence

$$(r_m - r_n) \int_{x_1}^{x_2} dx g V_m V_n = 0.$$

As r_m and r_n are supposed not to be the same, it follows that

$$\int_{x_1}^{x_2} dx g V_m V_n = 0.$$

Since we have

$$\int dx g V_m V_n = \frac{k}{r_m - r_n} \left(V_m \frac{dV_n}{dx} - V_n \frac{dV_m}{dx}\right),$$

it appears that when $m = n$,

$$\int_{x_1}^{x_2} dx g V_n^2 = \frac{0}{0};$$

the real value is

$$\int_{x_1}^{x_2} dx g V_n^2 = - V_n \frac{d}{dr_n}\left(k \frac{dV_n}{dx} + h_2 V_n\right) \text{ when } x = x_2;$$

as may be deduced by the usual method for evaluating indeterminate functions.

It is to be observed that the equation (3) involves an equation to determine r, which equation may be written as

$$F(r) = 0.$$

Poisson* has shewn that this equation has an infinite number of real and unequal roots, for the demonstration of which proposition I must refer to the works cited below.

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* *Bulletin de la Société Philomatique*, 1828. *Théorie de la Chaleur*, p. 178.

The function V is of great importance in the theory of heat, and the investigation of its properties has formed the subject of several elaborate memoires by MM. Sturm and Liouville. See *Journal de Mathématiques*, Tome I. pages 106, 253, 269, 373, and Tome II. p. 16.

(22) Let Y_m and Z_n be integral and rational functions of μ, $(1-\mu^2)^{\frac{1}{2}}$, $\cos\omega$ and $\sin\omega$ determined by the equations

$$\frac{d}{d\mu}(1-\mu^2)\frac{dY_m}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2Y_m}{d\omega^2}+m(m+1)Y_m=0,$$

$$\frac{d}{d\mu}(1-\mu^2)\frac{dZ_n}{d\mu}+\frac{1}{1-\mu^2}\frac{d^2Z_n}{d\omega^2}+n(n+1)Z_n=0,$$

then will

$$\int_{-1}^{+1}d\mu\int_0^{2\pi}d\omega\, Y_m Z_n=0,$$

so long as m and n are different.

Multiply both equations by $(1-\mu^2)$, and assume

$$(1-\mu^2)\frac{d}{d\mu}=\frac{d}{dt}, \text{ when they become}$$

$$\frac{d^2Y_m}{dt^2}+\frac{d^2Y_m}{d\omega^2}+m(m+1)(1-\mu^2)Y_m=0 \ldots (1),$$

$$\frac{d^2Z_n}{dt^2}+\frac{d^2Z_n}{d\omega^2}+n(n+1)(1-\mu^2)Z_n=0 \ldots (2).$$

Multiply (1) by $Z_n dt\,d\omega$ and (2) by $Y_m dt\,d\omega$, subtract (2) from (1) and integrate with respect to t and ω. Then transposing, and observing that $(1-\mu^2)\,dt=d\mu$, we have

$$\{m(m+1)-n(n+1)\}\int d\mu\int d\omega\, Y_m Z_n$$

$$=\iint dt\,d\omega\left\{Y_m\frac{d^2Z_n}{dt^2}-Z_n\frac{d^2Y_m}{dt^2}\right\}+\iint dt\,d\omega\left\{Y_m\frac{d^2Z_n}{d\omega^2}-Z_n\frac{d^2Y_m}{d\omega^2}\right\}.$$

Now if we effect the integration of the first term of the right hand side with respect to t, it becomes

$$\int d\omega\left\{Y_m\frac{dZ_n}{dt}-Z_n\frac{dY_m}{dt}\right\}=\int d\omega\,(1-\mu^2)\left\{Y_m\frac{dZ_n}{d\mu}-Z_n\frac{dY_m}{d\mu}\right\}.$$

In taking the limits from $t = -\infty$ to $t = +\infty$, or from $\mu = -1$ to $\mu = +1$, the part under the sign of integration vanishes, in consequence of the factor $1 - \mu^2$; hence on integrating with respect to ω from 0 to 2π we find that the first term of the right hand side of the equation is equal to zero. In the same way, on effecting the integration with respect to ω of the second term of the right hand side of the equation, we find it to become

$$\int dt \left\{ Y_m \frac{dZ_n}{d\omega} - Z_n \frac{dY_m}{d\omega} \right\},$$

which vanishes on taking it between the limits $\omega = 0$ and $\omega = 2\pi$, because Y_m and Z_n are supposed to be rational and integral functions of $\sin\omega$ and $\cos\omega$. Hence on integrating with respect to t and taking it between the limits $t = -\infty$ and $t = +\infty$, or $\mu = -1$ and $\mu = +1$, the second term of the right hand side also vanishes; therefore

$$\{m(m+1) - n(n+1)\} \int_{-1}^{+1} d\mu \int_0^{2\pi} d\omega \, Y_m Z_n = 0.$$

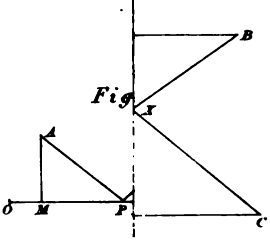
So long as m is different from n this involves the condition that

$$\int_{-1}^{+1} d\mu \int_0^{2\pi} d\omega \, Y_m Z_n = 0.$$

The functions Y_m and Z_n are known by the name of Laplace's Functions, that mathematician having been the first who studied their properties and pointed out their utility in the calculation of attractions. For the investigation of other remarkable theorems relating to these functions the reader is referred to the *Mécanique Céleste*, Liv. III., or to O'Brien's *Mathematical Tracts*. Mr Murphy has applied to the treatment of these functions a new and very remarkable analysis, which will be found in the introduction to his *Elementary Principles of the Theory of Electricity*.

THE END.

Fig



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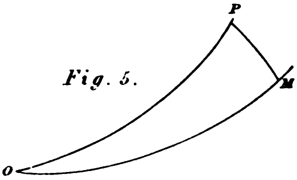
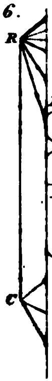
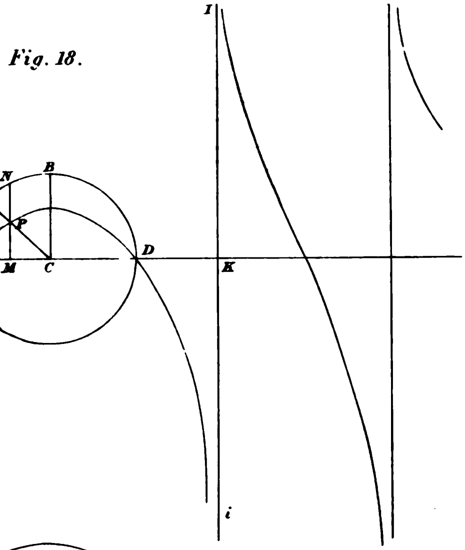


Fig. 18.

Fig. 6.





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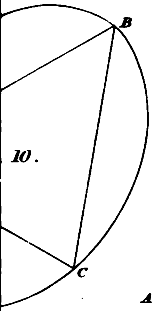

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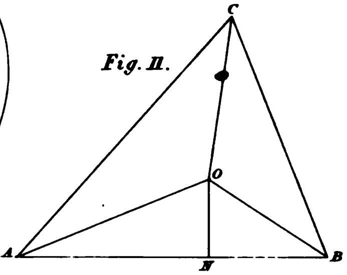

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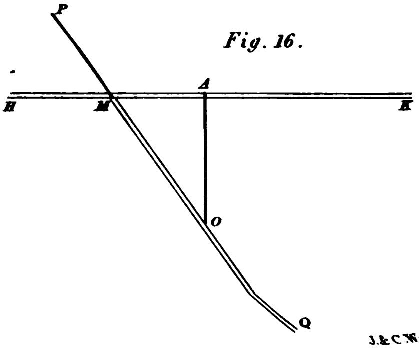

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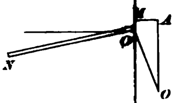

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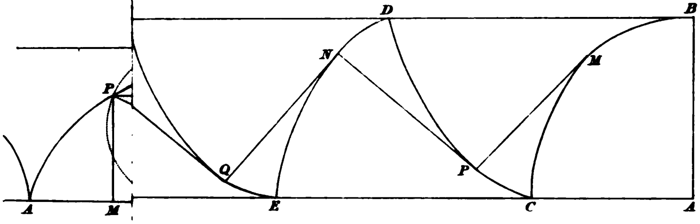
Fig. 21.



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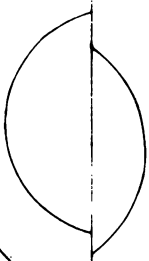

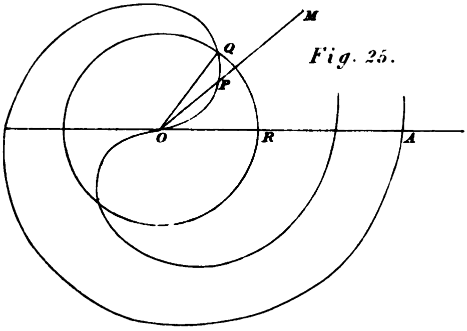

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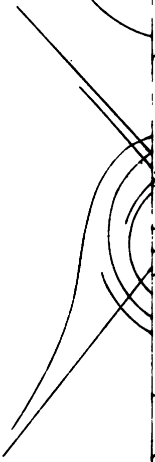

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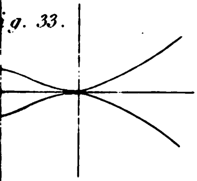

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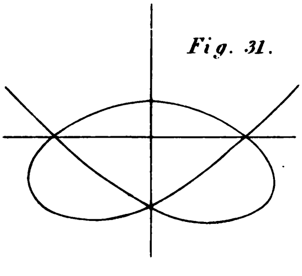


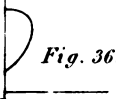
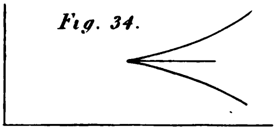
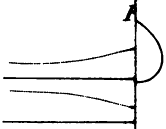


Fig. 36.

Fig. 34.





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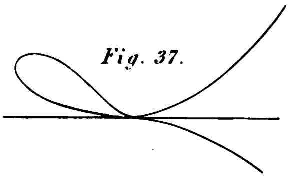

Fig. 3

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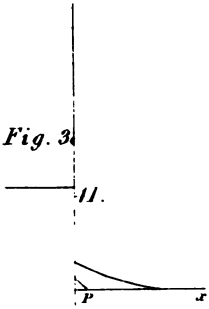


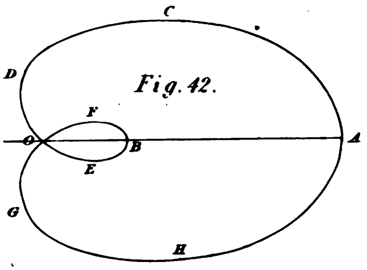
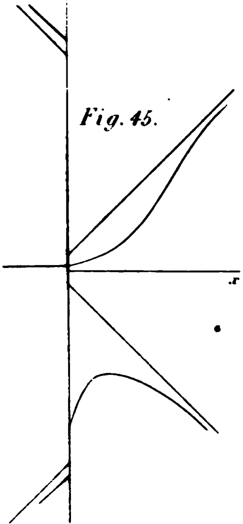


Fig. 42.

Fig. 45.



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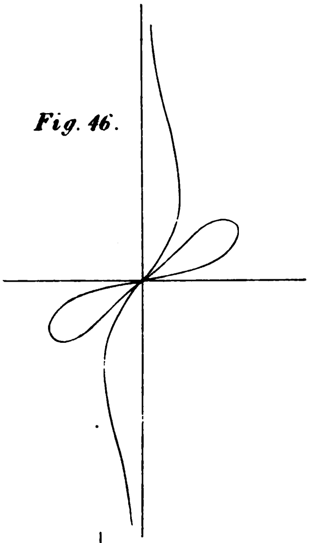

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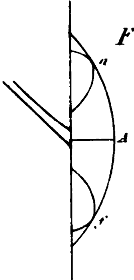

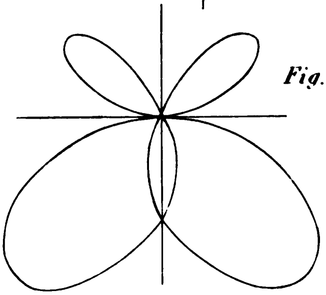

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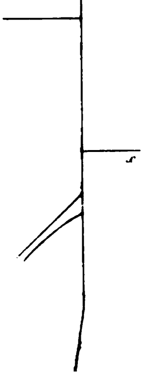

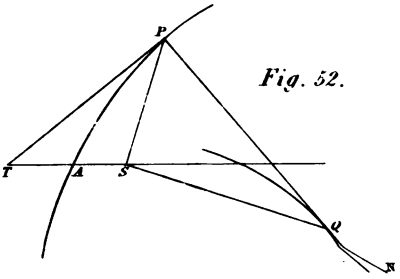

Fig. 52.

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